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MECHANICAL GEOMETRY.

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AN APPLICATION TO GEOMETRY OF SOME PROPOSITIONS IN STATICS.

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PREFACE.

Our knowledge of Geometry has been greatly increased by the application of Algebra and the Infinitesimal Calculus to that science. I have endeavoured, in the following pages, to demonstrate that *elementary* Statics can also be advantageously applied to Geometry, thinking that a complete separation of Pure from Applied Mathematics ought no more to be insisted upon than the separation of Geometry from Analysis.

This book will recommend itself to Problem Makers, and to those who want to satisfy themselves quickly as to the truth of certain geometrical statements. Some may also think that the road to a solution, short, and free from ambiguity, is often clearly pointed out by *the new methods;* and these, by comparison, will occasionally be found preferable to the use of Pure Geometry, of Co-ordinates, and of Abridged Notation. Originality of solution alone can be expected in subjects so hackneyed as the properties of straight lines and circles in one plane, to which this volume is confined. Some of the illustrating examples, however, are new.

In the *equations of points* (Chapter I.), and *of forces* (Chapter II.), I have extended the meaning of the sign of equality; in them it means *equivalence.* In Chapter II. and the following ones, multiplication often signifies *juxtaposition* of the capital letters.

I have explained and exemplified, in the first eight chapters, some of the processes by which equations of forces, of distances, and of areas, can be derived from equations of points. In Chapers II., VI., IX., X. will be found *the method of the separation of the letters* which define either a force, or a distance, or an area. The magnitude of a line is easily ascertained by the processes of Chapters VII. and VIII. Chapters XI. and XII. contain the application of the preceding ones to the Areal Equations of the Straight Line and Circle. An equation involving one variable only, can be obtained, representing a straight line when of the first degree, a conic when of the second degree, a cubic when of the third, and so on. Moreover, most of the propositions in this book are actually true for a System of Points in Space, the others being readily made so, by changing in their enunciations, right lines into planes, areas into volumes, and circles into spheres. These and other further applications of the Mechanical Geometry I have in manuscript.

LONDON, *February*, 1869.

CONTENTS.

MECHANICAL GEOMETRY.

CHAPTER I.

NOTATION, WITH ITS APPLICATION TO STRAIGHT LINES, TRIANGLES, QUADRILATERALS, POLYGONS AND CIRCLES IN ONE PLANE.—RATIOS OF SEGMENTS OF INTERSECTING LINES.—THREE OR MORE LINES THROUGH ONE POINT.—THREE OR MORE POINTS IN ONE RIGHT LINE.

1. NOTATION. Let the equation,

$$(a+b+c+d+\text{etc.})\ \mathrm{P}=a\mathrm{A}+b\mathrm{B}+c\mathrm{C}+d\mathrm{D}+\text{etc.} \quad (1)$$

mean that P is the centre of parallel forces: a, at the point A, b at B, c at C, etc.; those forces which act in one direction having the same sign, those acting in opposite direction having the opposite sign. Then $a+b+c+d+$etc. is the resultant of these parallel forces.

The equation $0=a\mathrm{A}+b\mathrm{B}+c\mathrm{C}+d\mathrm{D}+\text{etc.}$ (2)

means that parallel forces a at the point A, b at B, c at C, etc., are in equilibrium. Then the algebraic sum of these forces must be zero, or

$$0=a+b+c+d+\text{etc.}$$

The equation (1) is to be read:

$a+b+c+d+$etc. at P equals a at A $+\ b$ at B $+\ c$ at C $+$ etc.

And the equation (2) :

zero equals a at $A+b$ at B $+c$ at $C+d$ at D + etc.

2. Since a system of forces may be replaced by their resultant, it follows that to or from each member of either of the equations (1) or (2), we may add or subtract one or more terms of the form l L, whose meaning is a force l at L parallel to the forces a at A, b at B, etc. Thus from (1)

$$(a+b+c+\text{etc.})\ P+lL-mM=aA+bB+cC+\text{etc.}+lL-mM$$

and from (2) $\qquad lL-mM=aA+bB+cC+\text{etc.}+lL-mM$

Whence it follows that the ordinary rule for transposing one term from one member to the other holds true for such equations as (1) and (2).

Again, since a force may be replaced by a system of forces of which it is the resultant :

If zero or $(a+b+c+\text{etc.})\ P=aA+bB+cC+\text{etc.}$

and if $\qquad (l+m)\ C=l\ L+mM$

then zero or

$$(a+b+c+\text{etc.})\ P=aA+bB+c\frac{lL+mM}{l+m}+\text{etc.}$$

If all the forces composing a system of parallel forces be multiplied or divided by the same quantity, the position of their centre is unaltered, and the magnitude of the resultant is multiplied or divided by that quantity. Hence from (1) we have

$$m\ (a+b+c+\text{etc.})\ P=ma\ A+mb\ B+mc\ C+\text{etc.}$$

$$\text{and } \frac{1}{m}(a+b+c+\text{etc.})\ P=\frac{a}{m}A+\frac{b}{m}B+\frac{c}{m}C+\text{etc.}$$

Also, if a system of parallel forces be in equilibrium, and each force be multiplied or divided by the same quantity, the new system is also in equilibrium. Hence

from (2) we have

$$0 = ma\ \mathrm{A} + mb\ \mathrm{B} + mc\ \mathrm{C} + \text{etc.}$$

$$\text{and } 0 = \frac{a}{m}\mathrm{A} + \frac{b}{m}\mathrm{B} + \frac{c}{m}\mathrm{C} + \text{etc.}$$

Having established the preceding propositions, and remembering that if equivalent systems of forces be added to, or subtracted from, equivalent systems of forces, the results are also equivalent, we may state that:

Equations of the form (1) *or* (2) *may be treated like ordinary simple equations.*

(2) may be brought to the form (1) by transposing one term. For instance, (2) gives

$$-a\mathrm{A} \text{ or } (b+c+\text{ etc.})\ \mathrm{A} = b\mathrm{B} + c\mathrm{C} + \text{ etc.}$$

similarly from (1) we get

$$0 = a\mathrm{A} + b\mathrm{B} + c\mathrm{C} + \text{ etc.} - (a+b+c+\text{ etc.})\ \mathrm{P}.$$

3. In the following pages, I shall repeatedly make use of the following theorem:—

If there be a system of parallel forces in one plane, then their algebraic sum, multiplied by the distance of their centre from any straight line AB in that plane, is equal to the sum of the products of each force by the distance of its point of application from the same line AB. These distances being measured either all perpendicularly to AB, or all parallel to another line CD, also in the plane of the forces. This operation I shall call,

in the first case, "*Taking distances from* AB;"
in the second case, "*Taking distances from* AB, *parallel to* CD."

Points in one Straight Line.

4. *If* G *be a point on the right line* AB, *then*

$$\mathrm{AB}\,.\,\mathrm{G} = \mathrm{GB}\,.\,\mathrm{A} + \mathrm{AG}\,.\,\mathrm{B}\,.$$

for parallel forces proportional to GB, GA applied at A and B, and acting in the same direction, must have G for their centre.

If H *be a point on* BA *produced, then*

$$AB \,.\, H = HB \,.\, A - HA \,.\, B \,.$$

for parallel forces proportional to HB, HA applied at A and B, and acting in opposite directions, will have their centre at H.

5. *If* ABC *be points in order in a right line, then*

$$\frac{A-B}{AB} = \frac{B-C}{BC}$$

for $$AC \,.\, B = AB \,.\, C + BC \,.\, A$$

or $$(AB+BC)\, B = AB \,.\, C + BC \,.\, A$$

or $$AB\,(B-C) = BC\,(A-B)$$

or $$\frac{A-B}{AB} = \frac{B-C}{BC}$$ (See 2.)

6. ***Meaning of*** G ***in the equation*** $2G = A + B$.

This equation is abridged from

$$(1+1)G = 1A + 1B$$

and means that G is the centre of two equal parallel forces acting at A and B in the same direction; therefore G is the middle point of AB.

Meaning of G ***in the equation*** $3G = A + 2B$.

This equation is abridged from

$$(1+2)G = 1A + 2B$$

and means that G is the centre of two parallel forces acting at A and B in the same direction, and bearing to one another the ratio of one to two. Hence G is the point of trisection of AB nearer to B.

Meaning of G *in the equation* $2G = 3A - B$.

This equation is abridged from

$$(3-1)G = 3A - 1B$$

and indicates that G is the centre of two parallel forces

acting at A,B in opposite directions, and bearing the ratio of three to one. Hence G lies on BA produced, and AG is half of AB.

7. *Meaning of the coefficients x, y in the equation*

$$(x+y)G=xA+yB.$$

This equation means that G is the centre of parallel forces x, y applied at A and B. Hence G is a point in AB. Also by statics

$$xGA=yGB \qquad \text{or } x:y=GB:GA.$$

$$\text{also } (x+y)\ GA=yBA \text{ and } (x+y)GB=xAB.$$

Meaning of x, y in the equation $(x-y)H=xA-yB$.

H is the centre of parallel forces x, y acting in opposite directions at the points A, B. Hence, if x be greater than y, H lies on BA produced, and $xHA=yHB$ or $x:y=HA:HB$.

Also $(x-y)HA=yBA$ and $(x-y)HB=xAB$.

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8. *Line harmonically divided.*

If we have simultaneously the equations,

$$(x+y)G=xA+yB, \quad (x-y)H=xA-yB,$$

Then AB being divided internally in G and externally in H in the same ratio, HAGB is a line harmonically divided.

9. *Harmonic Pencil.*

If diverging lines SA, SC, SB, SD cut a straight line ACBD harmonically, they will cut every other straight line EGFH harmonically.

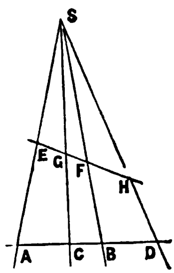
Since ACBD is harmonically divided,

let	$(x+y)C=xA+yB$	(1)
then	$(x-y)D=xA-yB$	(2)
Let	$(p+q)E=pS+qA$	(3)
and	$(l+m)F=lS+mB$	(4)

Getting the values of A and B from (3) and (4), and substituting in (1), therefore

$$(x+y)\mathrm{C}=\frac{x}{q}\left\{(p+q)\mathrm{E}-pS\right\}+\frac{y}{m}\left\{(l+m)\mathrm{F}-l\mathrm{S}\right\}$$

or $mq(x+y)\mathrm{C}+(mxp+lyq)\mathrm{S}=mx(p+q)\mathrm{E}+qy(l+m)\mathrm{F}$.



This equation means that parallel forces $mq(x+y)$ at C, $mxp+lyq$ at S, are equivalent to two forces parallel to them, viz., $mx(p+q)$ at E and $qy(l+m)$ at F. Hence the centre of these two systems of parallel forces must coincide, but the centre of the first is on CS, and that of the second on EF; therefore the common centre of the two systems is G, the intersection of CS and EF; and

$$\{mx(p+q)+qy(l+m)\}\,\mathrm{G}=mx(p+q)\mathrm{E}+qy(l+m)\mathrm{F} \quad (5)$$

Similarly to find H, which is the intersection of EF, SD, we shall eliminate A, B between (2), (3), (4); then

$$(x-y)\mathrm{D}=\frac{x}{q}\left\{(p+q)\mathrm{E}-p\mathrm{S}\right\}-\frac{y}{m}\left\{(l+m)\mathrm{F}-l\mathrm{S}\right\}$$

whence as above

$$\{mx(p+q)-qy(l+m)\}\,\mathrm{H}=mx(p+q)\mathrm{E}-qy(l+m)\mathrm{F} \quad (6)$$

from (5) and (6) EGFH is a line harmonically divided, See Article 8.

10. *Line geometrically divided.*

If $\quad (x+y)\mathrm{B}=x\mathrm{A}+y\mathrm{C} \quad (1)$

and $\quad (x+y)\mathrm{C}=x\mathrm{A}+y\mathrm{D} \quad (2)$

then $\mathrm{AC}^2=\mathrm{AB}\,.\,\mathrm{AD}$ *or* ABCD *is a straight line geometrically divided beginning with* A.

For from (1) $(x+y)AB=yAC$
from (2) $(x+y)AC=yAD$
multiplying crosswise and reducing, $AC^2=AB\ .\ AD$.

11. If ABCD be a line harmonically divided, and E the middle point of AC, then EBCD is geometrically divided.

By hypothesis, see 8, $(x+y)B=xA+yC$ (1)
and $(x-y)D=xA-yC$ (2)
and, see 6, $2E=A+C$ (3)

Eliminate A between (1) (3)

$$(x+y)B=2xE+(y-x)C \quad (4)$$

Between (2) and (3) eliminate A.

$$(x-y)D=2xE-(x+y)C$$
$$(x+y)C=2xE+(y-x)D \quad (5)$$

From (4) and (5) EBCD is a line geometrically divided. See 10.

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12. *Condition that three points should be in one right line.*

If $0=aA+bB+cC$, *then* A, B, C *are in one right line,* for then $-aA=bB+cC$, which (see 1), means that A is the centre of parallel forces b at B, c at C; but that centre lies on BC, or BC produced; therefore A, B, C, are in one straight line. Also $0=a+b+c$.

13. *Conditions that three straight lines should pass through one point.*

If $aA+bB=cC+dD=eE+fF$, *then the lines* AB, CD, EF, *intersect in one point.*

For the equation $aA+bB=cC+dD$ means that the parallel forces a at B, b at B are equivalent to the forces c at C, d at D parallel to them; hence the centres of these two systems must coincide; but the centre of a at A, b at B is some point on AB, and that of c at C,

d at D is some point on CD; therefore the common centre must be G, the intersection of AB, CD, so that

$$(a+b)\,G=aA+bB, \text{ and } (c+d)\,G=cC+dD;$$

also $$a+b=c+d.$$

Similarly the centre of $aA+bB$ coincides with that of $eE+fF$, but the latter is on EF therefore EF passes through G; hence the three lines AB, CD, EF, pass through one point, and $a+b=c+d=e+f$.

14. *If a point* D *is known to lie on* AB, *and* C *is without* AB, *and if* $(a+b+c)\,D=aA+bB+cC$, *then* c *is necessarily zero.*

15. *If* $a+b+c+\text{etc.}=0$, *then the point*

$$aA+bB+cC, \text{ etc., } \textit{is at infinity};$$

for this point is the point

$$-(b+c+\text{etc.})A+bB+cC+\text{etc.}$$

Let P be the point $bB+cC+$etc., then the point $aA+bB+cC+$etc. is also the point $-(b+c+\text{etc.})\,A+(b+c+\text{etc.})\,P$; that is, is the centre of equal finite parallel forces, acting at A, P, in opposition, which centre we know by statics to be at infinity.

TRIANGLE.

16. *Middle points of the sides of the triangle* ABC.

Let D, E, F be the middle points of BC, CA, AB; then, see Article 6,

$$2D=B+C,\ 2E=C+A,\ 2F=A+B$$

17. The bisectors AD, BE, CF of the sides of the triangle ABC meet in one point, and trisect each other; for the above equations give

$$2D+A=2E+B=2F+C=A+B+C$$

hence, see 13, the bisectors AD, BE, CF meet in one point G such that

$$3G=A+B+C$$

also, see 6, G is a point of trisection of each bisector; the bisectors therefore trisect each other.

18. *Centre of gravity of the area of the triangle ABC.*

By statics, this centre of gravity is the common intersection G of the bisectors of the sides of ABC, and

$$3G=A+B+C$$

19. *Feet of the bisectors of the interior and exterior angles of the triangle ABC.*

Let AD, BE, CF be the bisectors of the internal angles, A,B,C, of the triangle ABC, meeting in D, E, F the sides BC, CA, AB.

then $\quad BC\,.\,D=CD\,.\,B+BD\,.\,C \quad$ See 4.

but $\quad BD:CD:BC=c:b:b+c \quad$ Euclid, Book 6.

hence $\quad (b+c)\,D=bB+cC \quad$ (1) see 2.

similarly $\quad (c+a)\,E=cC+aA \quad$ (2)

and $\quad (a+b)\,F=aA+bB \quad$ (3)

Let AD′, BE′, CF′ be the bisectors of the external angles A, B, C of the triangle ABC, meeting in D′, E′, F′ the sides BC, CA, AB.

then $\quad BC\,.\,D'=BD'.\,C-CD'.\,B$

but $\quad BD':CD':BC'=c:b:c-b. \quad$ Euclid, Book 6.

hence $\quad (b-c)\,D'=bB-cC \quad$ (4)

similarly $\quad (c-a)\,E'=cC-aA \quad$ (5)

and $\quad (a-b)\,F'=aA-bB \quad$ (6)

20. From (1) and (4), BDCD′ is a line harmonically divided. See 8.

21. *Common intersection* O *of the bisectors of the interior angles of the triangle* ABC.

From (1), (2), (3), $(b+c)D+aA=(c+a)\,E+bB$

$=(a+b)$ $F+cC=aA+bB+cC$, which means, see 13, that AD, BE, CF, or the bisectors of the interior angles of the triangle, meet in one point O; such that

$$(a+b+c)O=aA+bB+cC$$

22. (4)+(5)+(6) gives

$$(b-c)\ D' + (c-a)\ E'+(a-b)\ F'=0$$

or the feet of the bisectors of the exterior angles of the triangle ABC lie in one right line. See 12.

23. Again finding the intersection of CF′ with BE′, which is done by eliminating A between (5) and (6),

$$(c-a)\ E'+bB=cC-(a-b)\ F'=\text{see (6) } cC-aA+bB$$
$$=-aA+(b+c)D.$$

Hence BE′, CF′, AD meet in one point, see 13, or the bisectors of two exterior angles, and the bisector of the internal third angle, meet in one point.

24. (2)—(3)+(4) give

$$(c+a)\ E-(a+b)\ F+(b-c)\ D'=0$$

or, see 12, the feet of the bisectors of two interior angles, and the foot of the bisector of the remaining third exterior angle lie in one right line.

25. *Feet of the perpendiculars of the triangle* ABC.

Let AD, BE, CF be drawn from A,B,C to meet BC, CA, AB at right angles in the points D,E,F:

Then $BC\,.\,D=CD\,.\,B+BD\,.\,C$, see 4,

but $CD=b\cos C$, $BD=c\cos B$,

therefore $CD:BD=\tan B:\tan C$,

hence $(\tan B+\tan C)D=\tan B\,.\,B+\tan C\,.\,C$ (1), see 2,

similarly $(\tan C+\tan A)E=\tan C\,.\,C+\tan A.A$ (2),

and $(\tan A+\tan B)F=\tan A\,.\,A+\tan B\,.\,B$ (3),

26. From (1), (2), (3)

$$\tan A\,.\,A+(\tan B+\tan C)D=\tan B\,.\,B+(\tan C+\tan A)\,E$$
$$=\tan C\,.\,C+(\tan A+\tan B)F,$$

hence, see 13, the perpendiculars AD, BE, CF of the triangle ABC meet in one point.

The common intersection of the perpendiculars of the triangle ABC *is the point.*

$(\tan A+\tan B+\tan C)P=\tan A.A+\tan B.B+\tan C.C.$

27. If EF,BC intersect in H, FD,CA in K, DE,AB in L, then H,K,L lie in one right line.

To find H the intersection of BC,EF eliminate A between (2) and (3),

then $(\tan A+\tan B)F-(\tan A+\tan C)E=$
$\tan B\,.\,B-\tan C\,.\,C=(\tan B-\tan C)H$, see 13.

similarly $\tan C\,.\,C-\tan A\,.\,A=(\tan C-\tan A)K$

and $\tan A\,.\,A-\tan B\,.\,B=(\tan A-\tan B)L$

adding the last three equations,

$(\tan B-\tan C)H+(\tan C-\tan A)K+(\tan A-\tan B)L=0$

Hence H,K,L lie in one right line, see 12.

Also BDCH is a line harmonically divided, see 8.

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28. *Any point within or without the triangle* ABC.

First. *Meaning of the coefficients* x,y,z *in the equation* $(x+y+z)G=xA+yB+zC$, *where* x,y,z *are all positive.*

This equation means that G is the centre of parallel forces x,y,z at the points A,B,C; hence, as x,y,z are all positive, G lies within the triangle ABC; then, drawing AA′,GG′ perpendiculars on BC, and taking distances from BC, see 3, we have $(x+y+z)GG'=xAA'$,

multiply by $\frac{1}{2}$ BC;

therefore $(x+y+z)$ area GBC$=x$ area ABC;

similarly $(x+y+z)$ area GCA$=y$ area ABC;

and $(x+y+z)$ area GAB$=z$ area ABC.

Hence $x : y : z =$ area GBC : area GCA : area GAB; or if G be the centre of parallel forces applied at A,B,C, these forces are proportional to the areas of the triangles GBC, GCA, GAB.

Next. *Meaning of the coefficients x,y,z in the equation* $(-x+y+z)G=-xA+yB+zC$, *where x,y,z are all positive.*

Let D be the point in BC, such that $(y+z)D=yB+zC$,

then $(-x+y+z)G=(y+z)D-xA$;

which means that G is the CG of parallel forces $y+z,x$ acting at D,A in opposite directions; hence by statics G lies on AD produced if $y+z$ is greater than x; or, if $y+z$ is greater than x, the point G lies in the indefinite space between BC, and AB produced and AC produced. But if $y+z$ is less than x, then G lies on DA produced; that is: when $y+z$ is less than x, G lies within the indefinite space between BA produced and CA produced. Also $x : y : z =$ area GBC : area GCA : area GAB.

29. Conversely. *If* G *be a point within the triangle* ABC, *then*

area ABC.G $=$ area GBC.A $+$ area GCA.B $+$ area GAB.C.

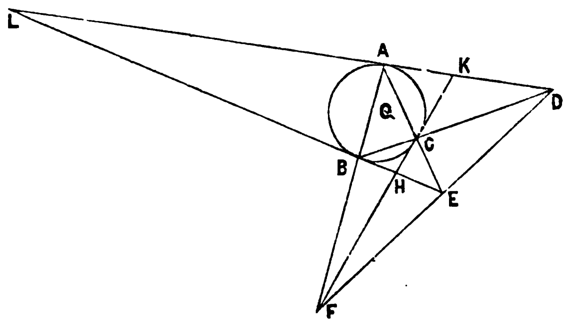
30. But, *If* G *lies without the triangle* ABC, *then in the preceding equation area* GBC *must be taken with the sign minus when* G *and* A *are on opposite sides of* BC, *area* GCA *with the sign minus when* G *and* B *are on opposite sides of* CA, *and area* GAB *with the sign minus when* G *and* C *are on opposite sides of* AB.

For instance, if G lies within the angle vertically opposite to BAC, then

area ABC.G $=$ area GBC.A $-$ area GCA.B $-$ area GAB.C.

Again, if G lies in the indefinite space between AC, BC produced and BA produced,
area ABC.G = area GBC.A — area GCA.B + area GAB.C.

The Circumscribed Circle of the Triangle ABC.



31. *Centre Q of the circle circumscribing the triangle* ABC.

Let q be the radius of this circle. Generally Q being any point.
Area ABC.Q = area QBC.A + area QCA.B + area QAB.C, see 29, 30.

here area QBC $= \frac{1}{2} q^2 \sin 2A$, area QCA $= \frac{1}{2} q^2 \sin 2B$, area QAB $= \frac{1}{2} q^2 \sin 2C$;

therefore area QBC : area QCA : area QAB $= \sin 2A : \sin 2B : \sin 2C$;

and, see 2, $(\sin 2A + \sin 2B + \sin 2C)Q =$

$$\sin 2A \,.\, A + \sin 2B \,.\, B + \sin 2C \,.\, C.$$
$$= 4 \sin A \sin B \sin C \,.\, Q.$$

32. G the CG of the triangle ABC, P the intersection of its perpendiculars, and Q the centre of its circumscribed circle lie in one right line; and G is the point of trisection of PQ nearer to Q.

In the triangle ABC, $\tan B + \tan C = \dfrac{\sin 2A}{2 \cos A \cos B \cos C}$;

or $\sin 2A : \sin 2B : \sin 2C = \tan B + \tan C : \tan C + \tan A : \tan A + \tan B$;

hence $2(\tan A + \tan B + \tan C)Q = (\tan B + \tan C)A + (\tan C + \tan A)B + (\tan A + \tan B)C$;

$= (\tan A + \tan B + \tan C)\ (A + B + C) - (\tan A . A + \tan B . B + \tan C . C)$;

$= 3(\tan A + \tan B + \tan C)G - (\tan A + \tan B + \tan C)P$

therefore, see 2,

$2Q = 3G - P$ or G,P,Q lie in one right line;

and since $3G = P + 2Q$, G is the point of trisection of PQ nearer to Q.

33. *If the tangent at* A *to the circumscribed circle meets* BC *in* D *then*

$$(b^2 - c^2)D = b^2B - c^2C$$

for $BC . D = BD . C - CD . B$ (1). See 4

but the equiangular triangles ABD, ACD give

$$AB : AC = BD : AD$$

hence $AB^2 : AC^2 = BD^2 : AD^2$

but $AD^2 = BD . CD$

therefore $AB^2 : AC^2 = BD : CD$ See 2

thus (1) becomes $(AB^2 - AC^2)\ D = AB^2 . C - AC^2 . B$

or $(c^2 - b^2)\ D = c^2C - b^2B$.

34. The tangents at A, B, C to the circle circumscribing ABC meet BC, CA, AB in points which are in one straight line—

we have $(b^2 - c^2)\ D = b^2B - c^2C$ (1)

similarly $(c^2-a^2)\ E=c^2C-a^2A$ (2)

and $(a^2-b^2)\ F=a^2A-b^2B$ (3)

add $(b^2-c^2)\ D+(c^2-a^2)\ E+(a^2-b^2)\ F=0$

or D, E, F lie in one right line. See 12.

35. *If* H *be the intersection of the tangents at* B *and* C *then*

$$(-a^2+b^2+c^2)\ H=-a^2A+b^2B+c^2C$$

between (2) and (3) eliminate A thus

$$(c^2-a^2)\ E+(a^2-b^2)\ F=c^2C-b^2B$$

hence $(c^2+b^2-a^2)\ H=(c^2-a^2)\ E+b^2B=c^2C-a^2A+b^2B$.

36. The lines AH, BK, CL meet in one point

for $(-a^2+b^2+c^2)\ H=-a^2A+b^2B+c^2C$

similarly $(a^2-b^2+c^2)\ K=a^2A-b^2B+c^2C$

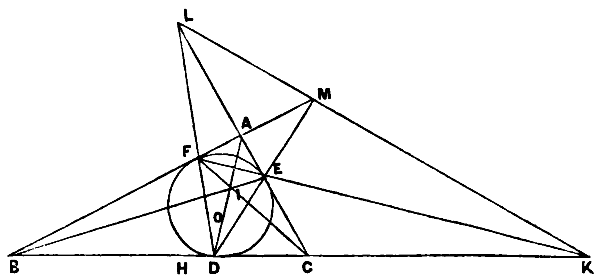
and $(a^2+b^2-c^2)\ L=a^2A+b^2B-c^2C$

whence $(-a^2+b^2+c^2)\ H+2a^2A=a^2A+b^2B+c^2C$

$=(a^2-b^2+c^2)\ K+2b^2B=(a^2+b^2-c^2)\ L+2c^2C$.

Thus AH, BK, CL meet in the point $a^2A+b^2B+c^2C$. See 13.

INSCRIBED CIRCLE OF THE TRIANGLE ABC.



37. *Centre* O *of the inscribed circle*

generally for any point O

area ABC . O=OBC . A+OCA . B+OAB . C

here area OBC$=\frac{1}{2}ar$, area OCA$=\frac{1}{2}br$, area OAB$=\frac{1}{2}cr$, where r is the radius of the inscribed circle; and a, b, c the sides BC, CA, AB, hence those areas are proportional to a, b, c, therefore

$$(a+b+c)\,\mathrm{O}=a\mathrm{A}+b\mathrm{B}+c\mathrm{C} \qquad (1).$$

38. Let AO meet BC in H, then from (1)
$(a+b+c)\,\mathrm{O}-a\mathrm{A}=b\mathrm{B}+c\mathrm{C}=(b+c)\,\mathrm{H}$. See 13
therefore AO : HO$=b+c:a$ and AH : OH$=a+b+c:a$.

39. *The points* D, E, F *of contact of the inscribed circle.*

See 4, BC . D$=$CD . B$+$BD . C
or $a\mathrm{D}=(s-c)\mathrm{B}+(s-b)\mathrm{C}$ where $2s=$perimeter of ABC

$$\text{or} \quad \left(\frac{1}{s-b}+\frac{1}{s-c}\right)\mathrm{D}=\frac{\mathrm{B}}{s-b}+\frac{\mathrm{C}}{s-c} \qquad (2)$$

$$\text{similarly} \quad \left(\frac{1}{s-c}+\frac{1}{s-a}\right)\mathrm{E}=\frac{\mathrm{C}}{s-c}+\frac{\mathrm{A}}{s-a} \qquad (3)$$

$$\text{and} \quad \left(\frac{1}{s-a}+\frac{1}{s-b}\right)\mathrm{F}=\frac{\mathrm{A}}{s-a}+\frac{\mathrm{B}}{s-b} \qquad (4)$$

Let r_1, r_2, r_3 denote the radii of the escribed circles of ABC which touch the sides BC, CA, AB respectively, then area ABC$=r_1\,(s-a)=r_2\,(s-b)=r_3\,(s-c)=rs$ and the equations (2) (3) (4) may be written

$$(r_2+r_3)\,\mathrm{D}=r_2\mathrm{B}+r_3\mathrm{C} \qquad (5)$$
$$(r_3+r_1)\,\mathrm{E}=r_3\mathrm{C}+r_1\mathrm{A} \qquad (6)$$
$$(r_1+r_2)\,\mathrm{F}=r_1\mathrm{A}+r_2\mathrm{B} \qquad (7).$$

40. From (5) (6) (7)
$r_1\mathrm{A}+(r_2+r_3)\mathrm{D}=r_2\mathrm{B}+(r_3+r_1)\mathrm{E}=r_3\mathrm{C}+(r_1+r_2)\mathrm{F}=$ $r_1\mathrm{A}+r_2\mathrm{B}+r_3\mathrm{C}$, or the lines AD, BE, CF joining the vertices of a triangle ABC with the points in which the inscribed circle touches the opposite sides meet in one point. See 13.

If I denote this point $(r_1+r_2+r_3)\,\mathrm{I}=r_1\mathrm{A}+r_2\mathrm{B}+r_3\mathrm{C}$ and area BIC : area CIA : area AIB$=r_1:r_2:r_3$.

41. The same equations give

$(r_3+r_1)\ E-(r_1+r_2)\ F=r_3C-r_2B=(r_3-r_2)\ K.$ See 13.

$(r_1+r_2)\ F-(r_2+r_3)\ D=r_1A-r_3C=(r_1-r_3)\ L$

$(r_2+r_3)\ D-(r_3+r_1)\ E=r_2B-r_1A=(r_2-r_1)\ M$

adding $0=(r_3-r_2)\ K+(r_1-r_3)L+(r_2-r_1)\ M,$

or the points K, L, M are in one right line. See 12, hence: The opposite sides of any triangle ABC, and of the triangle formed by joining the points of contact of its inscribed circle, intersect in three points lying in one right line.

Also $LM : MK=r_3-r_2 : r_1-r_3$

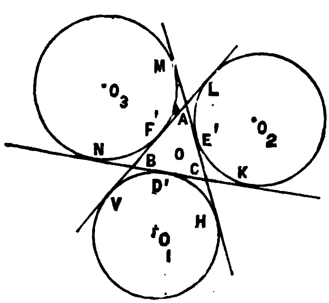
$EK : FK=r_1+r_2 : r_1+r_3.$

42. Since $(r_2+r_3)\ D=r_2B+r_3C$

and $(r_3-r_2)\ K=r_3C-r_2B$

therefore BDCK is a line harmonically divided.

ESCRIBED CIRCLES.



43. *Centres* O_1, O_2, O_3 *of the Escribed Circles.*

Let $(x+y+z)\,O_1 = xA+yB+zC.$

Take distances from BC, CA, AB respectively.

Then $(x+y+z)r_1 = x\left(-\frac{2\text{ area ABC}}{a}\right)$

$$(x+y+z)r_1=y\left(\frac{2\text{ area ABC}}{b}\right)$$

$$(x+y+z)r_1=z\left(\frac{2\text{ area ABC}}{c}\right)$$

Hence $\frac{x}{-a}=\frac{y}{b}=\frac{z}{c}=\frac{x+y+z}{-a+b+c}$

And therefore $(-a+b+c)O_1=-aA+bB+cC$
similarly $(a-b+c)O_2=aA-bB+cC$
and $(a+b-c)O_3=aA+bB-cC$

44. Now $(a+b+c)O=aA+bB+cC$

Eliminating A, B, C between the last four equations,

$$sO=(s-a)O_1+(s-b)O_2+(s-c)O_3 \text{ or } \frac{O}{r}=\frac{O_1}{r_1}+\frac{O_2}{r_2}+\frac{O_3}{r_3}$$

Hence area $O_1O_2O_3 : OO_2O_3 : OO_3O_1 : OO_1O_2=s : s-a : s-b : s-c$. See 28.

45. *Points of contact of the Escribed Circles.*

They are easily proved to be

$$aD'=(s-b)B+(s-c)C$$
$$bH=sC-(s-b)A$$
$$cV=sB-(s-c)A$$
$$aK=sC-(s-a)B$$
$$bE'=(s-c)C+(s-a)A$$
$$cL=sA-(s-c)B$$
$$aN=sB-(s-a)C$$
$$bM=sA-(s-b)C$$
$$cF'=(s-a)A+(s-b)B$$

46. $aD'+(s-a)A=bE'+(s-b)B=cF'+(s-c)C=(s-a)A+(s-b)B+(s-c)C$; or see 13.

Those lines meet in a point which join each vertex of a triangle with the point in which the corresponding escribed circle touches the opposite side.

Let U be this common intersection, then

$sU=(s-a)A+(s-b)B+(s-c)C=s(A+B+C)-(aA+bB+cC)=3sG-2sO$. See 18 and 37. Therefore $U=3G-2O$, hence U lies on OG produced, and $GU=2OG$. See 7.

47. The intersections P, R, S of (BC, E′F′), (CA, F′D′), (AB, D′E′) lie in one right line, and

$$0=(b-c)P+(c-a)R+(a-b)S.$$

48. AD', BH, CV meet in the point
$$-(s-b)(s-c)A+s(s-b)B+s(s-c)C.$$
49. VH, BC, $E'F'$ meet in a point.

50. The intersections of (VH, BC), (LK, CA), (MN, AB) lie in one straight line.

51. INSCRIBED AND ESCRIBED CIRCLES.

D, E, F being the points of contact of the inscribed circle. See 39.

(i.) The middle point of DD' coincides with that of CD.

For $aD=(s-c)B+(s-b)C$ and
$$aD'=(s-b)B+(s-c)C.$$
Adding $a(D+D')=a(B+C)$ or $D+D'=B+C$, hence see 6 and 13.

(ii.) AD, BM, CL meet in the point
$$-sA+(s-c)B+(s-b)C.$$
(iii.) The middle points of BO_1, AO_2, CF lie in one right line.

(iv.) AD' is parallel to the line joining the centre of the inscribed circle with the middle point of BC.

For $aD'=(s-b)B+(s-c)C$.

Therefore $a(A-D')=aA-(s-b)B-(s-c)C=$
$$aA+bB+cC-s(B+C).$$
Or $a(A-D')=2s(O-\frac{B+C}{2})$
$$=2s(O-\text{middle point } P \text{ of } BC).$$
Take distances from the line OP, then the distances of A and D' are equal and of the same sign, hence AD' is parallel to OP.

(v.) $6G=D+E+F+D'+E'+F'$, also
$$6G=V+H+K+L+M+N.$$

Circumscribed, Inscribed, and Escribed Circles.

52. OO_1 *and the arc* BC *of the circumscribed circle bisect each other.*

Let D be the middle point of the arc BC of the circumscribed circle, then AD bisects the angle BAC, and the points O, D, O_1 lie in it.

Let $D=xO+(1-x)O_1$, where x remains to be determined. Take distances from BC; then q denoting the radius of the circumscribed circle,

$$q-q\cos A=x(-r)+(1-x)r_1$$

$$\text{or } 2q\sin^2\frac{A}{2}=-4xq\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$

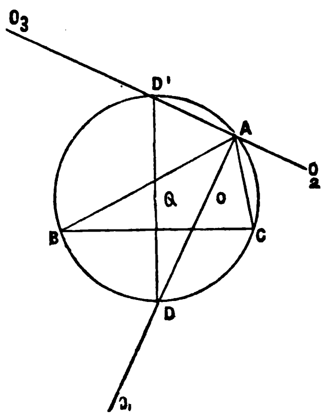
$$+4q(1-x)\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$$

$$\cos\frac{B+C}{2}=-2x\cos\frac{B-C}{2}+2\cos\frac{B}{2}\cos\frac{C}{2}$$

whence $x=\frac{1}{2}$ and therefore $2D=O+O_1$

53. O_2O_3 *is bisected by the circumference of the circumscribed circle.*

Let O_2O_3 meet the circumscribed circle in A and D′,



then AD′ is the bisector of the angle between BA and CA produced, but AD is the bisector of the adjacent angle BAC, hence DAD′ is a right angle; therefore DD′ is a diameter of the circumscribed circle; and since the arcs BD, CD are equal, DD′ is perpendicular on BC.

$$\text{Let } D'=xO_2+(1-x)O_3.$$

Take distances from BC, therefore

$$q+q\cos A=xr_2+(1-x)r_3$$

$$2q\cos^2\frac{A}{2}=4qx\sin\frac{B}{2}\cos\frac{A}{2}\cos\frac{C}{2}$$

$$+4(1-x)q\sin\frac{C}{2}\cos\frac{A}{2}\cos\frac{B}{2}$$

$$\sin\frac{B+C}{2}=2x\sin\frac{B-C}{2}+2\sin\frac{C}{2}\cos\frac{B}{2}$$

hence $x=\frac{1}{2}$ and therefore $2D'=O_2+O_3$

54. *If* Q *be the centre of the circumscribed circle*

$$4Q=O+O_1+O_2+O_3$$

for Q, D, D′ being the bisections of DD′, OO_1, O_2O_3, therefore $2Q=D+D'$, $2D=O+O_1$, $2D'=O_2+O_3$ hence $4Q=O+O_1+O_2+O_3$.

(i.) If the line joining any two of the four points O, O_1, O_2, O_3 be bisected in R, and that joining the other two in S, then Q bisects RS.

(ii.) Q lies on the line joining any one of the four points O, O_1, O_2, O_3 with the centre of gravity of the triangle formed by the other three.

(iii.) K the centre of gravity of $O_1\,O_2\,O_3$ lies on OQ produced, for $4Q=O+O_1+O_2+O_3$ and $3K=O_1+O_2+O_3$ therefore $4Q-O=3K$.

(iv.) L the centre of gravity of the triangle formed by joining the bisections D, E, F of the arcs BC, CA,

AB of the circumscribed circle is the point of trisection of QO nearer to Q.

for $3L=D+E+F$, therefore $6L=2D+2E+2F$

$6L=O+O_1+O+O_2+O+O_3=2O+(O+O_1+O_2+O_3)$

$=2O+4Q$, $3L=O+2Q$, hence, etc. See 6.

55. W *being the centre of the circle about* $O_1\,O_2\,O_3$ *then* $W=2Q-O$; for, see 31, (- - -) $W=\sin 2O_2\,O_1\,O_3.\,O_1+\sin 2O_3\,O_2\,O_1.\,O_2+\sin 2O_1\,O_3\,O_2.\,O_3$ but the angle $O_2\,O_1\,O_3$ is the complement of $\frac{A}{2}$, and therefore

$\sin 2O_2\,O_1\,O_3=\sin A$, and so on

hence $(a+b+c)\,W=aO_1+bO_2+cO_3$

$=s(O_1+O_2+O_3)-\{(s-a)O_1+(s-b)O_2+(s-c)O_3\}$

$=s(4Q-O)-sO$. See 54 and 44

$2W=4Q-O-O$ or $W=2Q-O$.

56. The points, O, L see 54 (iv.), Q, K see 54 (iii.), W see 55, are points in order in one right line and

$$OL:LQ:QK:KW=2:1:1:2.$$

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57. Centre G′ of Gravity of the Perimeter of the Triangle ABC.

Let D, E, F be the bisections of the sides BC, CA, AB; then (see 1)

$$(a+b+c)G'=aD+bE+cF=a\frac{B+C}{2}+b\,\frac{C+A}{2}+c\,\frac{A+B}{2}$$

therefore $2(a+b+c)\,G'=(b+c)\,A+(c+a)\,B+(a+b)C$

$=(a+b+c)\,(A+B+C)-(aA+bB+cC)$

or $2G'=3G-O$. See 18 and 37;

hence the CG of the perimeter of ABC lies on OG produced, at G′ so that $GG'=\frac{1}{2}OG$.

58. Again since $a=2EF$, $b=2FD$, $c=2DE$, then, see 1,

$2(EF+FD+DE)G'=2EF.D+2FD.E+2DE.F$

therefore $(EF+FD+DE)G'=EF.D+FD.E+DE.F$ which means that the CG of the perimeter of ABC is the centre of the circle inscribed in the triangle DEF. See 37.

59. The centre of gravity of the area of the triangle DEF whose angular points are the bisections of the sides of ABC coincides with the CG of ABC.

NINE POINTS CIRCLE.

60. It is easily shown that the circle through the feet of the perpendiculars of the triangle ABC, also passes through the middle points of BC, CA, AB and of PA, PB, PC where P is the intersection of the above mentioned perpendiculars. Hence the name of *Nine Points Circle.* Its radius is obviously half the radius q of the circumscribed circle.

61. *The centre Q' of the nine points circle is the point* $2Q'=3G-Q$; for Q' being the centre of the circle about the bisections D, E, F of the sides BC, CA, AB, therefore see 31.

$(\text{Sin } 2D+\sin 2E+\sin 2F)\, Q'=\sin 2D\,.\,D+\sin 2E.E+\sin 2F\,.\,F$; but the angle $D=A$, $E=B$, $F=C$; also $2D=B+C$, $2E=C+A$, $2F=A+B$, therefore

$$(\sin 2A+\sin 2B+\sin 2C)Q'$$
$$=\sin 2A\frac{B+C}{2}+\sin 2B\frac{C+A}{2}+\sin 2C\frac{A+B}{2}$$
$$=(\sin 2A+\sin 2B+\sin 2C)\frac{A+B+C}{2}$$
$$-\tfrac{1}{2}(\sin 2A\,.\,A+\sin 2B\,.\,B+\sin 2C\,.\,C)$$

or $2Q'=3G-Q$, see 31, and Q' lies on QG produced, so that $2Q'G=QG$.

62. $4Q'=3G+P$. See 18 and 26.
for as in 61,

$$4 \sin A \sin B \sin C \,.\, Q' = \sin 2A\frac{B+C}{2} + \sin 2B\frac{C+A}{2} + \sin 2C\frac{A+B}{2}$$

$$= \sin A \cos (B-C) \,.\, A + \sin B \cos (C-A) \,.\, B + \sin C \cos (A-B) \,.\, C$$

$$= \sin A \sin B \sin C (A+B+C) + \cos A \cos B \cos C (\tan A \,.\, A + \tan B \,.\, B + \tan C \,.\, C)$$

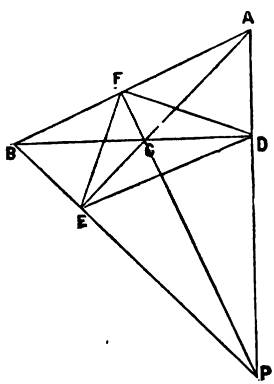
$$= 3 \sin A \sin B \sin C . G + \sin A \sin B \sin C . P$$

whence $4Q' = 3G + P$ or Q' is the point of quadrisection of PG nearest to G.

63. The four points Q, G, Q′, P lie in order in one right line, and $QG : GQ' : Q'P = 2 : 1 : 3$.

Triangle Formed by Joining the Feet of the Perpendiculars of the Triangle ABC.

64. The common intersection of the perpendiculars of the triangle ABC is the centre of the circle inscribed in the triangle formed by the feet of the perpendiculars of ABC when ABC is acute angled, but is the centre of one of the escribed circles of that triangle, when ABC is obtuse angled.



Proof when ABC is obtuse-angled at C.

Let D, E, F be the feet of the perpendiculars of ABC.

The centre of the circle touching DE and FD, FE produced is the point — DE.F + FD.E + EF.D. See 43,

that is the point c cos C.F+b cos B . E+a cos A.D
(since DE=$-c$ cos C)
or the point cos C(a cos B . A+b cos A . B)
+cos B(c cos A.C+a cos C.A)+cos A (b cos C.B+c cos B.C)
or the point

$$\frac{a}{\cos A}A+\frac{b}{\cos B}B+\frac{c}{\cos C}C$$

or the point tan A . A+tan B . B+tan C . C
that is the intersection P of the perpendiculars of ABC
See 26.

Lines drawn from the vertices of the Triangle ABC through the same Point.

65. L being any point within or without a triangle ABC; if AL, BL, CL meet BC, CA, AB in D, E, F. Then DB . EC . FA=DC . EA . FB.

Supposing L within the triangle ABC, and denoting by x, y, z quantities proportional to the areas of BLC. CLA, ALB, then

$$(x+y+z)\ L=xA+yB+zC$$

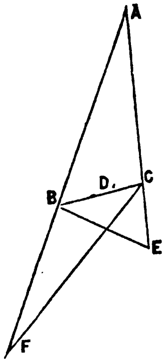
therefore $(x+y+z)$ L$-x$A$=y$B$+z$C$=(y+z)$ D See 13.

hence yBD$=z$CD See 7.
similarly zCE $=x$AE
and xAF$=y$BF

multiplying the last three equations, and dividing by xyz, the required result is obtained.

66. Conversely: If D, E, F be points on BC, CA, AB, or if one of them, such as D, lie on BC, and the other two E, F on CA produced, AB produced; and if DB . EC . FA=DC . EA . FB, then AD, BE, CF meet in one point.

Taking the second case, we have



BC . D=DC . B+BD . C (1)

CA . E=AE . C—CE . A (2) See 4.

AB . F=AF . B—BF . A (3)

To find H, the intersection of AD with BE, eliminate C between (1) and (2), then

(......)H=BC.AE.D—BD.CE.A (4)

To find K, the intersection of AD, CF, eliminate B between (1) and (3)

(......) K=BC . AF . D—DC.BF.A (5)

H and K will coincide, if

$$\frac{BC.AE}{BC.AF}=\frac{BD.CE,}{DC.BF,}$$

or if DB . EC . FA=DC . EA . FB

which is the case.

67. If through a given point L, within a triangle ABC, lines be drawn from the angles to the opposite sides, and the points of section be joined, the three first drawn lines are harmonically divided.

Let $(x+y+z)L=xA+yB+zC$

then $(x+y+z)\ L—zA=yB+zC=(y+z)\ D(1)$

similarly $zC+xA=(z+x)\ E$

and $xA+yB=(x+y)\ F$

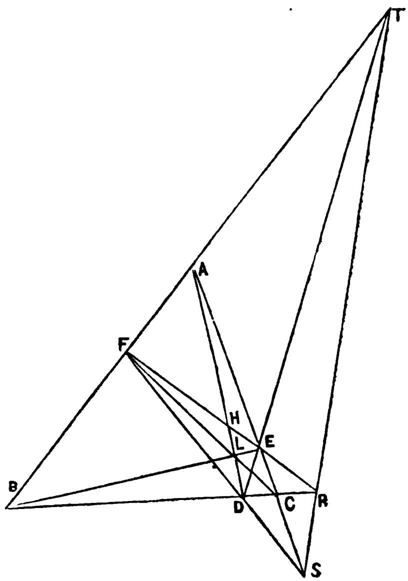
To find H, the intersection of AD, EF, eliminate B,C between the last three equations, then

$(z+x)\ E+(x+y)\ F=2xA+(y+z)\ D=$(etc.) H

using equation (1)

$$(2x+y+z)\ H=(x+y+z)\ L+xA \qquad (2)$$

from (1) and (2) AHLD is a line harmonically divided. See 8.



68. From the angles of ABC of a triangle, lines are drawn through a point L to meet the opposite sides in D, E, F respectively. FE, FD, DE are produced to meet BC, CA, AB in R, S, T. To prove that R, S, T, lie on one straight line.

Let $(x+y+z)\,L=xA+yB+zC$

then $(x+y+z)\,L-yB=xA+zC=(x+z)\,E$ See 13

similarly $xA+yB=(x+y)\,F$

To find R, the intersection of EF with BC, eliminate A between the last two equations;

then $(x+y)\,F-(x+z)\,E=yB-zC=(y-x)R$ See 13

similarly $zC-xA=(z-x)\,S$

and $xA-yB=(x-y)\,T$

by adding the last three equations we get

$$0=(y-z)\ \mathrm{R}+(z-x)\ \mathrm{S}+(x-y)\ \mathrm{T}$$

hence, see 12, the points R, S, T are in one right line.

69. *Conversely :* If DEF be a triangle inscribed in ABC, and if the opposite sides of the two triangles meet in points R, S, T lying in one right line, then AD, BE, CF meet in a point, and BFAT is a line harmonically divided.

Let
$$(x+y)\ \mathrm{D}=x\mathrm{B}+y\mathrm{C} \qquad (1)$$
$$(p+q)\ \mathrm{E}=p\mathrm{C}+q\mathrm{A} \qquad (2)$$
$$(r+s)\ \mathrm{F}=r\mathrm{A}+s\mathrm{B} \qquad (3)$$

suppose D, E, F to lie on BC, CA, AB, then we may suppose also x, y, z, p, q, r, s to be all positive.

To find R, the intersection of EF, BC, eliminate A between (2) and (3)

then $(qs-pr)\ \mathrm{R}=qs\mathrm{B}-pr\mathrm{C}$ (4)

similarly $(sy-rx)\ \mathrm{S}=sy\mathrm{C}-rx\mathrm{A}$ (5)

and $(qy-px)\ \mathrm{T}=qy\mathrm{A}-px\mathrm{B}$ (6)

Let e, f, g be the perpendiculars from A, B, C, on the right line RST

in (4) take distances from RST, then $o=qsf-prg$

similarly $o=syg-rxe$

and $o=qye-pxf$

Whence the condition $y^2s^2q^2=p^2r^2x^2$, and $ysq=prx$, (7) since x, y, p, q, r, s are all positive.

To find the intersection of AD with BE eliminate C between (1) and (2), and that intersection is

$$p(x+y)\mathrm{D}+qy\mathrm{A}$$

similarly the intersection of AD with CF is the point,

$$s(x+y)\mathrm{D}+rx\mathrm{A}$$

the two intersections coincide if

$\dfrac{p(x+y)}{s(x+y)}=\dfrac{qy}{rx}$ or if $rxp=qys$ which is true. See (7).

Again from (7) $qy:px=r:s$, hence substituting in (6)

$$(r-s)T=rA-sB \quad (8)$$

from (3) and (8) BFAT is a line harmonically divided. See 8.

70. D being any point, and A', B', C' any points in AD, BD, CD, to prove that the intersections E, F, G of (BC, $B'C'$), (CA, $C'A'$), (AB, $A'B'$) lie in one right line.

Let $D=xA'+(1-x)A=yB'+(1-y)B=zC'+(1-z)C$

Then $yB'-zC'=(1-z)C-(1-y)B=(y-z)E$. See 13

similarly $zC'-xA' \qquad =(z-x)F$

and $\qquad xA'-yB' \qquad =(x-y)G$

adding the last three equations

$$0=(y-z)E+(z-x)F+(x-y)G$$

or E, F, G lie in one right line. See 12.

71. If ABC be a given triangle, P any given point, and AD the 4th harmonic to AB, AP, AC intersect BC in D; BE the 4th harmonic to BC, BP, BA intersect CA in E; OF the 4th harmonic to CA, CP, CB intersect AB in F, then DEF lie in one right line. Ferrers, "Trilinear Coordinates," page 28.

Produce AP, BP, CP to meet BC, CA, AB in D', E', F'

Let $(x+y+z)P=xA+yB+zC$

then $(x+y+z)P-xA=yB+zC=(y+z)D$

whence since $BD'CD$ is a line harmonically divided

$$(y-z)D=yB-zC$$

similarly $\qquad (z-x)E=zC-xA$

and $\qquad (x-y)F=xA-yB$

and therefore $(y-z)D+(z-x)E+(x-y)F=0$

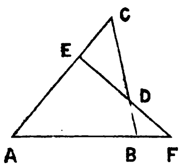
that is D, E, F lie in one right line. See 12.

72. From the angles A, B, C of a triangle, lines are

drawn through a point D, to meet the opposite sides in E, F, G respectively; FG, GE, EF are produced to meet BC, CA, AB respectively in P, Q, R; to prove that the lines BQ, CR, AE meet in a point, as also CR, AP, BF and AP, BQ, CG. Walton, page 46.

Straight Line Cutting the Sides of a Triangle or of a Polygon.

73. If a straight line intersects two sides AC, BC of a plane triangle in E, D and the base produced in F, to prove that AF . BD . CE = AE . BF . CD



here BC . D = CD . B + BD . C. See 4
and CA . E = AE . C + CE . A
and AB . F = AF . B — BF . A (1)
between the first two equations eliminate C, then

BC.AE.D—CA.BD.E = CD.AE.B—CE.BD.A = (....)F (2) see 13,

from (1) and (2) AF : BF = CD.AE : CE.BD, hence, etc.

74. D, E, F are points either all on the sides produced of the triangle ABC; or one of them on one side produced the other two on the sides themselves; then if AF . BD . CE = AE . BF . CD, the points D, E, F lie in one right line.

Take the case of the figure above, then

$$BC . D = CD . B + BD . C \quad (1)$$
$$CA . E = AE . C + CE . A \quad (2)$$
$$AB . F = AF . B - BF . A \quad (3).$$

The elimination of C between (1) and (2) gives

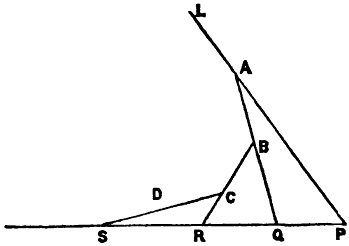
$$BC . AE . D - CA . BD . E = CD . AE . B - CE . BD . A$$
$$= \frac{AF . BD . CE}{BF} B - CE . BD . A$$

(making use of the condition in question)

$$=\frac{CE \cdot BD}{BF}(AF \cdot B - BF \cdot A) = \frac{CE \cdot BD}{BF} AB \cdot F$$

hence (see 12) D, E, F lie in one right line.

75. *Sides of a plane polygon intersected by a right line.*



Let ABCD......L be the plane polygon, PQRS......the transversal

We have

$$AQ \cdot B = AB \cdot Q + BQ \cdot A$$
$$BR \cdot C = BC \cdot R + CR \cdot B$$
$$CS \cdot D = CD \cdot S + DS \cdot C$$
$$\cdots\cdots\cdots\cdots\cdots$$
$$\cdots\cdots\cdots\cdots\cdots$$
$$LP \cdot A = AL \cdot P + AP \cdot L$$

Take distances from the transversal, and let a, b, c,...l denote the distances of A, B, C,......L from it.

Then (see 3)

$$AQ \cdot b = BQ \cdot a$$
$$BR \cdot c = CR \cdot b$$
$$CS \cdot d = DS \cdot c$$
and so on
$$LP \cdot a = AP \cdot l$$

Whence multiplying and reducing

$$AQ\,.\,BR\,.\,CS\ldots\ldots LP=BQ\,.\,CR\,.\,DS\ldots AP\,.$$

Mean Point.

76. *Definition.—If equal parallel forces acting in the same direction be applied at each of a series of n points* A, B, C,...L, *the centre* G *of these parallel forces is called the mean point of* A, B, C,...L

$$nG=A+B+C+\ldots\ldots+L.$$

Example.—The mean point of a triangle ABC is its centre of gravity,

for it is $\dfrac{A+B+C}{3}$ and therefore the centre of gravity.

Quadrilateral.

77. *Mean point of a Quadrilateral* ABCD.

It is G such that: $4G=A+B+C+D$

Let E, F be the bisections of AB, CD

then $2E=A+B$, $2F=C+D$

therefore $2G=E+F$, or the mean point of a quadrilateral bisects the line joining the middle points of two opposite sides. Hence, the lines bisecting opposite sides of a quadrilateral bisect each other.

Again, if M, L be the middle points of AC, BD, so that $2M=A+C$, and $2L=B+D$, then $2G=L+M$, or the mean point of a quadrilateral bisects the line joining the bisections of its two diagonals.

Hence the lines bisecting the opposite sides of a quadrilateral, and its diagonals, meet in one point and bisect each other.

Again, if P be the CG of the triangle ABC, then

$$3P=A+B+C\text{; therefore } 4G=3P+D$$

or the mean point of a quadrilateral is the point of quadrisection of the line joining one vertex with the CG of the triangle formed by the other three, nearest to that CG.

78. *If* ABCD *be a quadrilateral, then*

$0=\text{area BCD}\,.\,A-CDA\,.\,B+DAB\,.\,C-ABC\,.\,D.$

Let $$(x+y+z)D=xA+yB+zC$$

Take distances from BC,

$$(x+y+z)\frac{2\text{ area BCD}}{BC}=x\frac{2\text{ area ABC}}{BC}$$

or $$(x+y+z)\text{ area BCD}=x\text{ area ABC} \qquad (1)$$

Taking distances from CA,

$$(x+y+z)\text{ area DAC}=-y\text{ area ABC} \qquad (2)$$

Taking distances from AB,

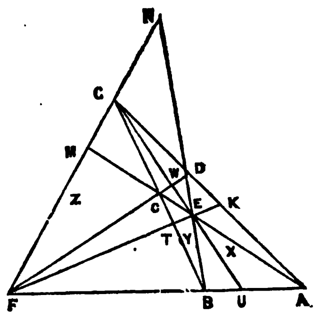
$$(x+y+z)\text{ area DAB}=z\text{ area ABC} \qquad (3)$$

from (1), (2), (3),

$$\frac{x}{\text{area BCD}}=\frac{y}{-\text{area DAC}}=\frac{z}{\text{area DAB}}=\frac{x+y+z}{\text{area ABC}}$$

whence $\text{area ABC}\,.\,D=BCD\,.\,A-CDA\,.\,B+DAB\,.\,C.$

79. *The properties of a quadrilateral.*



Let ABCD be any quadrilateral; let α, β, γ, δ, be proportional to the areas of the triangles BCD, CDA, DAB, ABC, then $0=\alpha A-\beta B+\gamma C-\delta D.$ (1) See 78. Let the diagonals AC, BD meet in E; and the opposite sides AB, CD in F; and the opposite sides BC, AD in G; also let AC, BD meet FG in M, N;

let GE meet CD, AB in W, U; let FE meet CB, AD in T,K;

$$\text{from (1)}\quad \alpha A+\gamma C=\beta B+\delta D=(\alpha+\gamma)E=(\beta+\delta)E \quad (2)$$

$$\text{and}\quad \alpha A-\beta B=\delta D-\gamma C=(\alpha-\beta)F=(\delta-\gamma)F \quad (3)$$

$$\alpha A-\delta D=\beta B-\gamma C=(\alpha-\delta)G=(\beta-\gamma)G \quad (4)$$

(i.) (3)—(4) give

$$\delta D-\beta B=(\delta-\gamma)F-(\beta-\gamma)G=(\delta-\beta)N \quad (5)$$

(3)+(4) give

$$\alpha A-\gamma C=(\delta-\gamma)F+(\beta-\gamma)G=(\alpha-\gamma)M \quad (6)$$

from (5) and (6), FMGN *is a line harmonically divided.*

(ii.) again (2)—(3) give

$$\gamma C+\beta B=(\beta+\delta)E-(\delta-\gamma)F=(\gamma+\beta)T \quad (7)$$

from (4) and (7), BTCG *is a line harmonically divided.* Similarly FBUA, FCWD are lines harmonically divided.

(iii.) *The points* N, T, W *are in one straight line.*

(2)—(4) give

$$(\beta+\delta)E-(\beta-\gamma)G=\delta D+\gamma C=(\delta+\gamma)W \quad (8)$$

(5)+(7) give

$$(\delta-\beta)N+(\gamma+\beta)T=\delta D+\gamma C=\text{therefore }(\delta+\gamma)W$$

hence N, T, W are in one right line.

Similarly MTU, UKN, MWK lie in right lines.

80. The middle points XYZ of the diagonals AC, BD, FG of a complete quadrilateral lie in one right line.

We have

$$2X=A+C \quad (9)$$

$$2Y=B+D \quad (10)$$

$$2Z=F+G \quad (11)$$

$$\text{also}\quad 0=\alpha A-\beta B+\gamma C-\delta D \quad (1)$$

between (11) and (3), (4), eliminate F and G

$$2(\delta-\gamma)(\beta-\gamma)Z=(\beta-\gamma)(\alpha A-\beta B)+(\delta-\gamma)(\alpha A-\delta D)$$

$$=\alpha(\beta-2\gamma+\delta)A-\beta(\beta-\gamma)B-\delta(\delta-\gamma)D$$

multiply (1) by an arbitrary quantity, λ, and subtract

the result from the last equation, then $2(\delta-\gamma)(\beta-\gamma)Z$ $=\alpha(\beta-2\gamma+\delta-\lambda)A-\beta(\beta-\gamma-\lambda)B-\lambda\gamma C-\delta(\delta-\gamma-\lambda)D$

Choose λ so that the coefficients of A and C are equal,

then $\qquad \alpha(\beta-\gamma+\delta)-\alpha\gamma-\alpha\lambda=-\lambda\gamma$

but $\qquad 0=\alpha-\beta+\gamma-\delta$, hence $\beta-\gamma+\delta=\alpha$

therefore $\qquad \alpha^2-\alpha\gamma=\lambda(\alpha-\gamma)$, hence $\lambda=\alpha$

whence

$$2(\delta-\gamma)(\beta-\gamma)Z=-\alpha\gamma(A+C)+\beta(\alpha+\gamma-\beta)B+\delta(\alpha+\gamma-\delta)D$$
$$=\beta\delta(B+D)-\alpha\gamma(A+C)$$

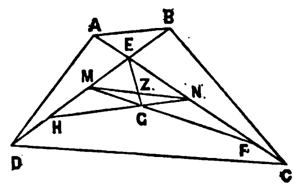
or $\qquad (\delta-\gamma)(\beta-\gamma)Z=\beta\delta Y-\alpha\gamma X$

hence X, Y, Z lie in one right line,

also $XZ:YZ=\beta\delta:\alpha\gamma$

$\qquad =$area CDA . area ABC : area BCD . area DAB.

81. *Centre of gravity of the area of the quadrilateral* ABCD.



Let G denote the required CG; let M, N be the bisections of BD, AC; E the intersection of AC, BD; Z the middle point of MN; along DB measure DH=BE; along CA measure CF=AE; join HN, FM intersecting in the required point G. This point G may also be found by producing EZ to G, so that $ZG=\frac{1}{3}EZ$.

(i.) Let α, β, γ, δ denote the areas BCD, CDA, DAB, ABC. The weight of the area ABCD is made up of the weights of the areas BCD, DAB; these weights are proportional to α, γ. Again, the weight of a triangular plate may be replaced by three weights at its corners, each one-third the weight of the plate. Hence the weight of the quadrilateral may be replaced by weights proportional to

$\frac{\alpha}{3}$ at B, $\frac{\alpha}{3}$ at C, $\frac{\alpha}{3}$ at D; $\frac{\gamma}{3}$ at D, $\frac{\gamma}{3}$ at A, $\frac{\gamma}{3}$ at B;

therefore $(\alpha+\gamma)\,G=\frac{\gamma}{3}A+\frac{\alpha+\gamma}{3}B+\frac{\alpha}{3}C+\frac{\alpha+\gamma}{3}D$

$3(\alpha+\gamma)G=(\alpha+\gamma)\,(B+D)+\gamma A+\alpha C$ (1)

but $0=\alpha A-\beta B+\gamma C-\delta D$ See 78. (2)

hence $(\alpha+\gamma)E=\alpha A+\gamma C$ whence $AE:CE=\gamma:\alpha$

but $CF=AE$, hence, $CF:AF=\gamma:\alpha$

hence $(\alpha+\gamma)F=\gamma A+\alpha C$ See 4, 7;

and (1) becomes $3G=2M+F$, or *the centre of gravity of the quadrilateral is the point of trisection of* MF *nearer to* M.

(ii.) similarly $3G=2N+H$, and G lies on NH and on MF

Therefore *the centre of gravity of the quadrilateral is the intersection of* MF *with* NH.

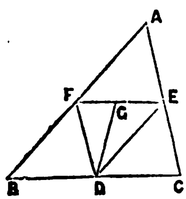
(iii.) $3(\alpha+\gamma)G=(\alpha+\gamma)(B+D)+\gamma A+\alpha C$ See (1)

similarly $3(\beta+\delta G=(\beta+\delta)(A+C)+\delta B+\beta D$

add $3(\alpha+\beta+\gamma+\delta)G=\{(\alpha+\beta+\gamma+\delta)(A+B+C+D)-(\alpha A+\beta B+\gamma C+\delta D)\}=\{4(\alpha+\beta+\gamma+\delta)Z-(\alpha A+\gamma C)-(\beta B+\delta D)\}=4(\alpha+\beta+\gamma+\delta)Z-(\alpha+\gamma)E-(\beta+\delta)E$ See(2)

hence $3G=4Z-E$, or G *lies on* EZ *produced and* $3GZ=EZ$.

82. *To find the centre of gravity of two uniform rods* AB, AC *having one common extremity* A.



Let D, E, F be the bisections of BC, CA, AB. Let G be the CG of the rods AB, AC, then

$(AB+AC)G=AB\,.\,F+AC\,.\,E$ See 1.

divide by 2, therefore

$(DE+DF)G=DE\,.\,F+DF\,.\,E$, hence,

see 19, the centre of gravity required is the foot of the bisector of the interior angle D of the triangle DEF.

83. *To find geometrically the centre of gravity of the perimeter of a quadrilateral* ABCD.

Bisect AB, BD, DA in E, F, G; draw EG, and the bisector of the angle EFG; these two lines will intersect in some point H, which is the centre of gravity of the rods AB, AD. See 82. Again, bisect DC, CB in K, L; join KL, and draw the bisector of the angle KFL to intersect KL in M, which is the CG of the rods BC, CD. Hence the weights of the sides may be replaced by two heavy particles, one at H and the other at M; therefore the required centre of gravity of the perimeter lies on HM.

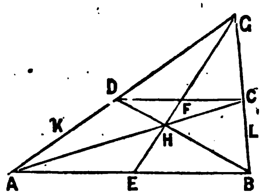
Again, bisect AC in N; join EL, and draw the bisector of the angle ENL to meet EL in O, which is the CG of the rods AB, BC. Join GK, and draw the bisector of the angle GNK to intersect GK in P, which is the CG of the rods AD, DC. Then the required centre of gravity of the perimeter of ABCD lies on OP; but it has been proved to lie on HM. Therefore it is the point Q of intersection of OP, HM.

Trapezium.—Parallel Lines.

84. *If* ABCD *be a trapezium, of which the parallel sides are* AB, DC, *then* $\frac{A-B}{AB}=\frac{D-C}{DC}$.

For ABCD being a quadrilateral, therefore

$$0 = \text{area } BCD.A - CDA.B + DAB.C - ABC.D.$$



But BCD, CDA, DAB, ABC are triangles having the same altitude, viz., the perpendicular distance between the parallel sides AB, DC, of the trapezium; hence these triangles are as their bases,

or $\quad 0 = CD.A - CD.B + AB.C - AB.D$

whence $\quad CD(A-B) = AB(D-C)$ and $\dfrac{A-B}{AB} = \dfrac{D-C}{DC}.$

85. Conversely: *If* $\dfrac{A-B}{x} = \dfrac{D-C}{y}$ *and* x, y *be both positive, then* AB, DC *are parallel lines pointing in the same direction, and* AB : DC $= x : y$.

The above equation gives $\quad A = B + \dfrac{x}{y}D - \dfrac{x}{y}C$

take distances from CD, let a, b be the distance of A, B from CD, then $a = b$, hence AB is parallel to CD.

Again, take distances from AD parallel to AB or CD, then

$$0 = AB - \frac{x}{y}CD$$

since x, y are both positive, AB, CD have the same sign; hence B,C are on the same side of AD; that is AB, DC are parallel lines pointing in the same direction.

86. The line joining the bisections K, L of the non-parallel sides AD, BC of a trapezium, is parallel to the other two sides and half their sum.

Here	$CD(A-B) = AB(D-C)$	(1)	See 84.
and	$2K = A + D$	(2)	
and	$2L = B + C$	(3)	See 6.

(2)—(3) give

$$2(K-L)=(A-B)+(D-C)=(A-B)+\frac{CD}{AB}(A-B)$$

using (1)

hence $AB\,(K-L)=\frac{1}{2}\,(AB+CD)\,(A-B)$; or KL is parallel to AB and equals $\frac{1}{2}\,(AB+CD)$. See 85.

87. The non-parallel sides AD, BC of a trapezium ABCD, and the line which bisects in E and F, the parallel sides AB, DC meet in one point.

Here $CD(A-B)=AB(D-C)$ (1)

$2E=A+B$ (2)

$2F=C+D$ (3).

Let G be the intersection of AD, BC; then from (1)

$$(CD-AB)G=CD.A-AB.D=CD.B-AB.C \quad (4)$$

CD (2)—AB(3) gives

$$\begin{aligned}2CD.E-2AB.F&=CD.A-AB.D+CD.B-AB.C\\&=2(CD-AB)G \qquad \text{using (4)}\end{aligned}$$

or EF passes through G.

88. The diagonals of a trapezium intersect in H on the line bisecting the parallel sides.

From (1) $(CD+AB)H=CD.A+AB.C=CD.B+AB.D$

therefore $2(CD+AB)H=CD(A+B)+AB(C+D)$

$$=2CD.E+2AB.F \quad (5)$$

hence H lies on EF.

89. EHFG is a line harmonically divided.

PARALLELOGRAM.

90. *If* ABCD *be a parallelogram, then*

$$0=A-B+C-D.$$

Generally for any quadrilateral ABCD

$$0=\text{area BCD.A}-\text{CDA.B}+\text{DAB.C}-\text{ABC.D}$$

here area BCD=area CDA=area DAB=area ABC

hence $0=A-B+C-D$.

This equation gives $A+C=B+D$, or the diagonals of a parallelogram bisect each other.

91. *Conversely.* If $0=A-B+C-D$, then ABCD is a parallelogram.

For the above equation gives $A+C=B+D$, or the middle points of AC, BD coincide, in E suppose; and the equal triangles EAB, ECD give AB equal and parallel to DC. Therefore ABCD is a parallelogram.

Example. The figure formed by joining the middle points E, F, G, H of the sides AB, BC, CD, DA of any quadrilateral, is a parallelogram.

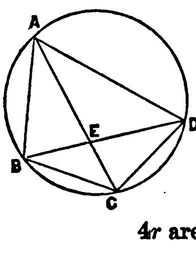
Here $2E=A+B, 2F=B+C, 2G=C+D, 2H=D+A$

therefore $2E-2F+2G-2H=0$, or EFGH is a parallelogram.

Quadrilateral Inscribed in a Circle.

92. *If* ABCD *be a quadrilateral inscribed in a circle, then*

$$0=\text{BC.CD.DB.A}-\text{CD.DA.AC.B}+\text{DA.AB.BD.C}-\text{AB.BC.CA.D}.$$



For any quadrilateral ABCD

$$0=\text{area BCD.A}-\text{CDA.B}+\text{DAB.C}-\text{ABC.D}$$

here, denoting by r the radius of the circle

$$4r \text{ area BCD}=\text{BC.CD.DB},$$

$$4r \text{ area CDA}=\text{CD.DA.AC, etc.}$$

hence $0=BC.CD.DB.A-CD.DA.AC.B+DA.AB.BD.C$
$-AB.BC.CA.D.$ (1)

93. *Corollary* 1. Generally when $0=aA+bB+cC+\text{etc.}$,
then $0=a+b+c+\text{etc.}$

therefore if ABCD be a quadrilateral inscribed in a circle
$0=BC.CD.DB-CD.DA.AC+DA.AB.BD-AB.BC.CA$
hence $AC:BD=BC.CD+DA.AB:CD.DA+AB.BC$
and $AB:CD=DA.AC-BC.BD:DA.DB-CA.CB$

94. *Corollary* 2. Let E be the intersection of the diagonals AC, BD, then from (1)

$$(\ldots)E=BC.CD.DB.A+DA.AB.BD.C$$

which reduces to

$$(\ldots)E=BC.CD.A+DA.AB.C$$

therefore $AE:CE=AB.AD:CB.CD$

95. If two opposite sides AB, CD of an inscribed quadrilateral be equal, so will the diagonals.

In (1) divide by AB or CD
hence

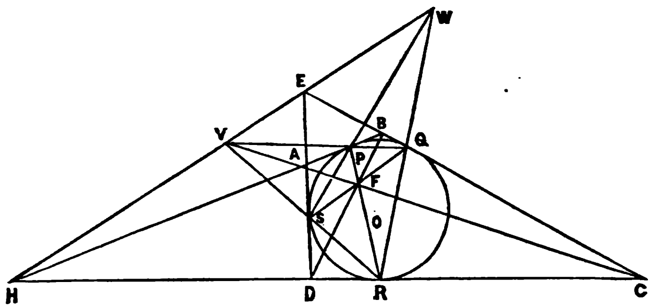
$$0=BC.DB.A-DA.AC.B+DA.BD.C-BC.CA.D$$

hence also $0=BC.DB-DA.AC+DA.DB-BC.CA$
therefore $AC(AD+BC)=BD(AD+BC)$
hence $AC=BD.$

96. If the two diagonals of an inscribed quadrilateral be equal, so will two opposite sides be equal.

This theorem is proved in the same way as the preceding.

Quadrilateral Circumscribing a Circle.



97. If a quadrilateral ABCD be circumscribed about a circle, the centre O of the circle lies on the line joining the middle points of the diagonals AC, BD.

Let BC, AD produced meet in E; since O is the centre of the escribed circle of the triangle EAB which touches the side AB and the sides EA, EB produced; therefore

$$(-AB+EA+EB)O=-AB.E+EA.B+EB.A \quad (1)$$

and since O is the centre of the circle inscribed in the triangle ECD,

$$(EC+CD+DE)O=EC.D+ED.C+CD.E \quad (2)$$

(2)—(1) gives $(AB+BC+CD+DA)O$

$$=EC.D+DE.C+(AB+CD)E-EB.A-EA.B;$$

but $\quad EC\,.\,B=EB\,.\,C+BC\,.\,E$

and $\quad ED\,.\,A=EA\,.\,D+AD\,.\,E.$

Subtract the last two equations from the preceding one, then $\quad (AB+BC+CD+DA)O$

$$=(EC-EA)(D+B)+(ED-EB)(C+A) \quad (3)$$

the coefficient of E vanishing, since $BC+AD=AB+CD$. The equation (3) shows that O lies on the line joining

the point B+D with the point C+A, that is on the line joining the middle points of the diagonals BD, AC, and divides it in the ratio of ED — EB : EC — EA.

Corollary. (AB+CD)O

$$=(BC+AD)O=(EC-EA)\frac{B+D}{2}+(ED-EB)\ \frac{A+C}{2}$$

similarly $$=(HC-HA)\frac{B+D}{2}+(HB-HD)\frac{A+C}{2}$$

where H is the intersection of AB, CD produced, hence

$$EC-EA=HC-HA=AP+CR \quad \text{and}$$
$$ED-EB=HB-HD=BQ+DS$$

98. Given a, b, c, d the tangents from the corners of a quadrilateral ABCD circumscribing a circle, to find its radius r.

$$\text{The angle } SOP=2\tan^{-1}\frac{a}{r}$$
$$POQ=2\tan^{-1}\frac{b}{r}$$
$$QOR=2\tan^{-1}\frac{c}{r}$$
$$ROS=2\tan^{-1}\frac{d}{r}$$

half the sum is $\pi=\tan^{-1}\frac{a}{r}+\tan^{-1}\frac{b}{r}+\tan^{-1}\frac{c}{r}+\tan^{-1}\frac{d}{r}$

whence

$$r^2(a+b+c+d)=abc+abd+acd+bcd.$$

99. To find the area of the triangle ABD in terms of a, b, c, d.

$$\text{Area ABD}=\tfrac{1}{2}\ AB.AD\ \sin A=\tfrac{1}{2}(a+b)(a+d)\sin POS$$
$$=(a+b)(a+d)\frac{ar}{a^2+r^2}$$

$$\text{area ABD}=\frac{a}{a+c}(a+b+c+d)^{\frac{1}{2}}(abc+abd+acd+bcd)^{\frac{1}{2}}$$

100. *Corollary* 1.

area BCD : area CDA : area DAB : area ABC

$$=\frac{c}{a+c}:\frac{d}{b+d}:\frac{a}{a+c}:\frac{b}{b+d}$$

101. *Corollary* 2. The area ABCD of the circumscribed quadrilateral being the sum of the areas of ABD, BCD is

$$(a+b+c+d)^{\frac{1}{2}}(abc+abd+acd+bcd)^{\frac{1}{2}}.$$

102. *If* ABCD *be a quadrilateral circumscribing a circle, and* a, b, c, d *be the tangents to this circle from* A, B, C, D, *then*

$$0=\frac{c}{a+c}\mathrm{A}-\frac{d}{b+d}\mathrm{B}+\frac{a}{a+c}\mathrm{C}-\frac{b}{b+d}\mathrm{D}. \qquad (1)$$

For ABCD being any quadrilateral,

$$0=\text{area BCD.A}-\text{CDA.B}+\text{DAB.C}-\text{ABC.D},$$

whence, using the preceding Corollary 1, we obtain the required result.

103. Let F be the intersection of the diagonals AC, BD,

then $$\frac{c\mathrm{A}+a\mathrm{C}}{c+a}=\mathrm{F}=\frac{d\mathrm{B}+b\mathrm{D}}{d+b} \qquad (2)$$

therefore $\mathrm{AF}:\mathrm{CF}=a:c.$

104. The diagonals of a circumscribed quadrilateral ABCD, and of the inscribed quadrilateral formed by joining the points P, Q, R, S of contact of AB, BC, CD, DA, intersect in one point.

From (2) $$\left(\frac{1}{a}+\frac{1}{c}\right)\mathrm{F}=\frac{\mathrm{A}}{a}+\frac{\mathrm{C}}{c}$$

$$\qquad (2 \text{ bis})$$

and $$\left(\frac{1}{b}+\frac{1}{d}\right)\mathrm{F}=\frac{\mathrm{B}}{b}+\frac{\mathrm{D}}{d}$$

add $\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)F=\frac{A}{a}+\frac{B}{b}+\frac{C}{c}+\frac{D}{d}$

$$=\left(\frac{A}{a}+\frac{B}{b}\right)+\left(\frac{C}{c}+\frac{D}{d}\right)$$

$$=\left(\frac{1}{a}+\frac{1}{b}\right)P+\left(\frac{1}{c}+\frac{1}{d}\right)R \quad (3)$$

also $$=\left(\frac{A}{a}+\frac{D}{d}\right)+\left(\frac{B}{b}+\frac{C}{c}\right)$$

$$=\left(\frac{1}{a}+\frac{1}{d}\right)S+\left(\frac{1}{b}+\frac{1}{c}\right)Q \quad (4)$$

From (3) F lies on PR, from (4) F lies on SQ; hence AC, BD, PR, QS meet in the same point F.

105. The lines PQ, RS, AC meet in one point.

Since $$\left(\frac{1}{a}+\frac{1}{b}\right)P=\frac{A}{a}+\frac{B}{b} \quad (5)$$

$$\left(\frac{1}{b}+\frac{1}{c}\right)Q=\frac{B}{b}+\frac{C}{c} \quad (6)$$

$$\left(\frac{1}{c}+\frac{1}{d}\right)R=\frac{C}{c}+\frac{D}{d} \quad (7)$$

$$\left(\frac{1}{d}+\frac{1}{a}\right)S=\frac{D}{d}+\frac{A}{a} \quad (8)$$

therefore (5)—(6) and (8)—(7) give

$$\left(\frac{1}{a}+\frac{1}{b}\right)P-\left(\frac{1}{b}+\frac{1}{c}\right)Q=\frac{A}{a}-\frac{C}{c}=\left(\frac{1}{d}+\frac{1}{a}\right)S-\left(\frac{1}{c}+\frac{1}{d}\right)R$$

or PQ, AC, RS meet in one point.

If V be that point $\left(\frac{1}{a}-\frac{1}{c}\right)V=\frac{A}{a}-\frac{C}{c}$ (9)

106. From (2 bis) and (9), VAFC is a line harmonically divided.

107. Similarly, D, F, B, W are in one right line, harmonically divided,

and $$\left(\frac{1}{b}-\frac{1}{d}\right)W=\frac{B}{b}-\frac{D}{d} \qquad (10)$$

108. The points H, V, E, W, are in one right line harmonically divided.

For from (1), the intersection E of AB, CD, is

$$(\ldots)E=\frac{cA}{c+a}-\frac{bD}{b+d}, \quad (\ldots)E=\frac{dB}{b+d}-\frac{aC}{c+a}$$

adding $$(\ldots)E=\frac{cA-aC}{c+a}+\frac{dB-bD}{b+d}$$

$$(\ldots)E=\frac{c-a}{c+a}V+\frac{d-b}{d+b}W \text{ (11), using (9) and (10)}$$

Again $$(\ldots)H=\frac{cA}{c+a}-\frac{dB}{b+d} \text{ and } (\ldots)H=\frac{bD}{d+b}-\frac{aC}{a+c}$$

adding $$(\ldots)H=\frac{cA-aC}{c+a}-\frac{dB-bD}{b+d}$$

or $$(\ldots)H=\frac{c-a}{c+a}V-\frac{d-b}{d+b}W \text{ (12), using (9) and (10).}$$

From (11) and (12) E, H lie on VW, and divide it harmonically.

---

109. *Mean point* G *of any hexagon* ABCDEF

$$6\,G=A+B+C+D+E+F=(A+B+C)+(D+E+F)$$
$$=3H+3K$$

where H, K are the CGs of the triangles ABC, DEF, therefore $$2G=H+K$$

or the mean point of any hexagon bisects the line joining the centre of gravity of the triangle formed by any three of the vertices of the hexagon, with the CG of the triangle formed by the other three.

Since the points ABCDEF may be divided into ten different pairs of triangles, by joining three of them to form one triangle, and the other three to form the other

triangle; it follows that ten lines, such as HK, intersect in the point G.

110. *Regular Polygon.*

Let ABCD...L be a regular polygon of n sides, O its centre. It is obvious that O is the mean point of the polygon, and therefore

$$nO = A + B + C + \ldots + L.$$

The same point O is also the centre of the circle inscribed in the polygon, of the circle described about it, and the centre of gravity of its area and perimeter.

111. The mean point of any polygon coincides with that of another polygon formed by joining the points which divide each side in the same ratio.

Let ABCD...L be the polygon, let n be the number of its sides, let A′B′C′D′... be the new polygon, also of n sides, such that A′ divides AB in a given ratio $x : y$, B′ divides BC in the same ratio $x : y$, and so on.

Let G, G′ be the mean points of the two polygons,

then $\quad nG = A + B + C + D + \ldots L$

$\quad\quad nG' = A' + B' + C' + D' + \ldots L'$

also $\quad (x+y)A' = yA + xB$

$\quad\quad (x+y)B' = yB + xC$

$\quad\quad (x+y)C' = yC + xD$

and so on.

$\quad\quad (x+y)L' = yL + xA$

add $\quad n(x+y)G' = (x+y)(A + B + C + \ldots + L)$

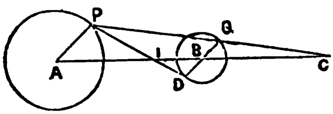
or $\quad nG' = A + B + C + \ldots + L = nG$

therefore $\quad G' = G.$

Circles.

112. *External centre of similitude of two circles.*

Let A, B, be the centres of the two circles; a, b, their radii. Let AP, BQ be two parallel radii of these circles pointing in the same direction, then PQ meets the line of centres AB in a fixed point C, which is called the external centre of similitude of the circles;



for PA, QB being parallel,

$$\frac{P-A}{a}=\frac{Q-B}{b} \qquad \text{See 84.}$$

$$\text{therefore } \frac{Q}{b}-\frac{P}{a}=\frac{B}{b}-\frac{A}{a}=\left(\frac{1}{b}-\frac{1}{a}\right)C \qquad (1)$$

but A,B are fixed points; a,b constants; hence C is a fixed point.

113. *Internal centre of similitude of two circles.*

If PA, BD be parallel radii, pointing in opposite directions, then PD meets AB in a fixed point I, called the internal centre of similitude of the two circles:

$$\text{for } \frac{P-A}{a}=\frac{B-D}{b}$$

$$\text{therefore } \frac{P}{a}+\frac{D}{b}=\frac{B}{b}+\frac{A}{a}=\left(\frac{1}{a}+\frac{1}{b}\right)I \qquad (2)$$

but A, B are fixed points, and a, b constants, hence I is a fixed point.

Corollary. From (1) and (2) the internal and external centres of similitude of two circles divide harmonically the line of centres.

114. The three external centres of similitude of three circles lie in one straight line. Two internal and one external centres of similitude always lie in one straight line.

Let A, B, C be the centres of the circles; a, b, c their radii; let A′, B′, C′ be the external centres of similitude of (B, C), (C, A), (A, B); let A″, B″, C″ be their internal centres of similitude:

then $$\left(\frac{1}{b}-\frac{1}{c}\right)A' = \frac{B}{b}-\frac{C}{c} \qquad (1)$$

$$\left(\frac{1}{c}-\frac{1}{a}\right)B' = \frac{C}{c}-\frac{A}{a} \qquad (2)$$

$$\left(\frac{1}{a}-\frac{1}{b}\right)C' = \frac{A}{a}-\frac{B}{b} \qquad (3)$$

$$\left(\frac{1}{b}+\frac{1}{c}\right)A'' = \frac{B}{b}+\frac{C}{c} \qquad (4)$$

$$\left(\frac{1}{c}+\frac{1}{a}\right)B'' = \frac{C}{c}+\frac{A}{a} \qquad (5)$$

$$\left(\frac{1}{a}+\frac{1}{b}\right)C'' = \frac{A}{a}+\frac{B}{b} \qquad (6)$$

(1) + (2) + (3) gives

$$\left(\frac{1}{b}-\frac{1}{c}\right)A'+\left(\frac{1}{c}-\frac{1}{a}\right)B'+\left(\frac{1}{a}-\frac{1}{b}\right)C'=0$$

or A′, B′, C′ lie in one right line.

(5)—(6) + (1) gives

$$\left(\frac{1}{c}+\frac{1}{a}\right)B''-\left(\frac{1}{a}+\frac{1}{b}\right)C''+\left(\frac{1}{b}-\frac{1}{c}\right)A'=0$$

or B″, C″, A′ lie in one right line.

Similarly C″, A″, B′ lie in one right line, and also A″, B″, C′.

115. AA″, BB″, CC″ meet in one point

$$\frac{A}{a}+\frac{B}{b}+\frac{C}{c}=(\ldots)\,L$$

AA′, BB′, CC″ meet in one point

$$\frac{A}{a}+\frac{B}{b}-\frac{C}{c}=(\ldots)\,M$$

L, M divide CC″ harmonically. See 8.

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116. *To find the centre of gravity of an arc of a circle.*

Let O be the centre of the circle; A, B, C any three points on the circumference; let 2α, 2β, 2γ, denote the circular measures of the angles BOC, COA, AOB.

The CG of the arc BC lies in the line joining O with the middle point of BC, let it divide that line as $x : 1-x$, then it will be

$$x\frac{B+C}{2}+(1-x)\,O$$

Similarly the CG of arc CA will be $y\dfrac{C+A}{2}+(1-y)\,O$

and that of the arc AB will be $z\dfrac{A+B}{2}+(1-z)\,O$

now O is the CG of the whole circumference:

$$\text{hence } (\alpha+\beta+\gamma)\,O=\alpha\left\{x\frac{B+C}{2}+(1-x)\,O\right\}$$

$$+\beta\left\{y\frac{C+A}{2}+(1-y)\,O\right\}+\gamma\left\{z\frac{A+B}{2}+(1-z)\,O\right\}$$

This gives, after reductions,

$$2(\alpha x+\beta y+\gamma z)O$$
$$=(\beta y+\gamma z)A+(\gamma z+\alpha x)B+(\alpha x+\beta y)C$$

but $(\sin 2\alpha+\sin 2\beta+\sin 2\gamma)O$

$$=\sin 2\alpha A+\sin 2\beta B+\sin 2\gamma C$$

hence

$$\frac{\beta y+\gamma z}{\sin 2\alpha}=\frac{\gamma z+\alpha x}{\sin 2\beta}=\frac{\alpha x+\beta y}{\sin 2\gamma}=.\,.\frac{2\alpha x}{\sin 2\beta+\sin 2\gamma-\sin 2\alpha}$$

$$=\frac{2\beta y}{\sin 2\gamma+\sin 2\alpha-\sin 2\beta}=\frac{2\gamma z}{\sin 2\alpha+\sin 2\beta-\sin 2\gamma}$$

or, since $\alpha+\beta+\gamma=\pi$

$$\frac{\alpha x}{\sin\alpha\cos\beta\cos\gamma}=\frac{\beta y}{\sin\beta\cos\alpha\cos\gamma}=\frac{\gamma z}{\sin\gamma\cos\alpha\cos\beta}$$

hence $x=\lambda\dfrac{\tan\alpha}{\alpha}$, where λ is some constant, for the position of the CG of the arc BC does not depend on that of A, and therefore on the magnitudes of β and γ.

Hence, if G be the CG of the arc BC,

$$G=\lambda\frac{\tan\alpha}{\alpha}\,\frac{B+C}{2}+\left(1-\lambda\frac{\tan\alpha}{\alpha}\right)O$$

Now, when the arc BC diminishes indefinitely, G coincides with $\dfrac{B+C}{2}$, hence, when α is indefinitely small,

$1-\lambda\dfrac{\tan\alpha}{\alpha}=0$, and $\dfrac{\tan\alpha}{\alpha}=1$, therefore $\lambda=1$ always.

And $G=\dfrac{\tan\alpha}{\alpha}\,\dfrac{B+C}{2}+\left(1-\dfrac{\tan\alpha}{\alpha}\right)O$

where 2α is the circular measure of the angle which the arc BC subtends at the centre.

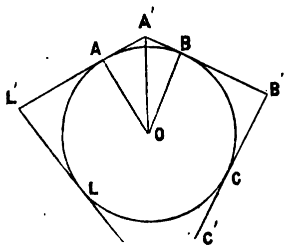
117. Similarly: *If* G′ *denotes the centre of gravity of the sector* BOC

$$G'=\frac{2}{3}\,\frac{\tan\alpha}{\alpha}\,\frac{B+C}{2}+\left(1-\frac{2}{3}\,\frac{\tan\alpha}{\alpha}\right)O$$

Cor. $3G'=2G+O$.

118. $A'B'C'D'\ldots L'$ *is a polygon circumscribing a circle, centre* O; A, B, C...L *are the points of contact of the sides* $L'A'$, $A'B'$, $B'C'$, *etc.; to prove*

$$(A'B'+B'C'+C'D'+\ldots+L'A')O$$
$$=L'A'.A+A'B'.B+B'C'.C+\ldots$$



For the circumference being made up of the arcs AB, BC, CD..., if we denote by 2α, 2β, 2γ, ... the circular measures of the angles AOB, BOC, etc.

$$(\alpha+\beta+\gamma+\ldots)O=\alpha\left\{\frac{\tan\alpha}{\alpha}\frac{A+B}{2}+\left(1-\frac{\tan\alpha}{\alpha}\right)O\right\}$$
$$+\beta\left\{\frac{\tan\beta}{\beta}\frac{B+C}{2}+\left(1-\frac{\tan\beta}{\beta}\right)O\right\}+\ldots$$

whence after reductions

$$2(\tan\alpha+\tan\beta+\tan\gamma+\ldots)O$$
$$=\tan\alpha\ (A+B)+\tan\beta\ (B+C)+\ldots\tan\lambda\ (L+A)$$
$$=(\tan\lambda+\tan\alpha)A+(\tan\alpha+\tan\beta)B+\text{etc.}$$

multiplying by the radius of the circle

$$(A'B'+B'C'+C'D'+\ldots+L'A')O$$
$$=(L'A+AA')A+(A'B+BB')B+\ldots, \text{ whence, etc.}$$

119. If a quadrilateral circumscribes a circle, the centre O of the circle lies in the line joining the middle points of the diagonals of the quadrilateral.

Using the figure of article 97, if a, b, c, d be the tangents from A, B, C, D to the circle, we have (see 118),

$$2(a+b+c+d)O$$
$$=(a+b)P+(b+c)Q+(c+d)R+(d+a)S$$
$$=bA+aB+bC+cB+cD+dC+dA+aD. \quad \text{See 4}$$
$$=(a+c)(B+D)+(b+d)(A+C)$$

hence O lies on the line joining $\frac{B+D}{2}$ with $\frac{A+C}{2}$

EXAMPLES.

[1]. If two triangles have a common centre of gravity and a common vertex, the extremities of their bases are the corners of a parallelogram.

[2]. O is any point; P, Q, R the centres of gravity of the triangles OBC, OCA, OAB; to prove that AP, BQ, CR meet in the mean point of the points A, B, C, O.

[3]. If the four points A, B, C, D be divided into pairs, viz. (AB, CD), (AC, BD), (AD, BC), and P be any point, then the three lines joining the centres of gravity of (PAB, PCD), (PAC, PBD), (PAD, PBC) meet in one point and bisect each other.

[4]. G being the CG of ABC, and D any point on BC (not BC produced), if DG be produced to E so that $GE=\frac{1}{2}GD$, then E is always within the triangle ABC.

[5]. $A'B'C'$ being the feet of the perpendiculars of ABC, Q the centre of its circumscribed circle,

$$(B'C'+C'A'+A'B')Q=B'C'.A+C'A'.B+A'B'.C.$$

[6]. If the inscribed circle touches BC in D, and DE be a diameter of that circle, then AE meets BC in a point F, such that BF=CD. (London University, 1865.)

[7]. If D, E, F be the points of contact of the inscribed circle, centre O; and DO meet EF in K, then FK : KE=AC : AB; and AK passes through the centre of gravity of ABC. Use 118.

[8]. If the third escribed circle touches AB in F′, the middle points of AO_2, BO_1, CF′ lie in one right line.

[9]. If the escribed circles touch BC, CA, AB in D, E, F, then O_1D, O_2E, O_3F meet in the centre of the circle about $O_1O_2O_3$. See 55.

[10]. If $(x+y+z)P=xA+yB+zC$ then $(x+y+z)\ P=(-x+y+z)D+(x-y+z)E+(x+y-z)F$ where D, E, F are the bisections of BC, CA, AB.

Hence area PEF : area PFD

=area ABC—2 PBC : area ABC—2 PCA.

[11]. The point $2sO'=aO_1+bO_2+cO_3$ lies on OQ produced, so that QO′=QO. See 44 and 54.

[12]. ABCD is a parallelogram, E a point on AB, such that $AE=\frac{1}{3}$ AB, DE cuts AC in F, to prove $AF=\frac{1}{4}$ AC.

[13]. ABCD is any quadrilateral, P any point; the bisectors of the angles APB, DPC meet AB, DC in E, F; the bisectors of the angles BPC, APD meet BC, AD in G, H; the bisectors of the angles BPD, APC meet BD, AC in K, L. To prove that EF, GH, KL meet in one point.

[14]. ABCDEF is any hexagon; G, L, H, K the centres of gravity of the triangles ABC, CDE, DEF, FAB; to prove that GLHK is a parallelogram.

[15]. P being any point in the plane of the triangle ABC; if the lines bisecting the angles BPC, CPA, APB meet BC, CA, AB in D, E, F; then AD, BE, CF meet in the point $\frac{A}{PA}+\frac{B}{PB}+\frac{C}{PC}$.

[16]. P being any point in the plane of the triangle ABC; if AP, BP, CP meet BC, CA, AB in D, E, F;

and if along CB or BC produced, CD′ be measured equal to BD, but in opposite direction; and along AC or CA produced AE′ be measured equal to CE, but in opposite direction; and along BA, or AB produced, BF′ be measured equal to AF, but in opposite direction; then AD′, BE′, CF′ meet in one point P′.

If $$(x+y+z)P=xA+yB+zC, \text{ then}$$
$$\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)P'=\frac{A}{x}+\frac{B}{y}+\frac{C}{z}.$$
The point P′ may be defined the *reciprocal* of P, then P is the reciprocal of P′. The points I (see 40) and U (see 46), are reciprocals.

[17]. In the preceding example, if AD″ be drawn to meet BC in D″, and making the angle CAD″=BAD, and so that AD, AD″ are both within, or both without the triangle ABC and on opposite sides of AB; and if BE″, CF″ be similarly drawn; then AD″, BE″, CF″ meet in one point P″.

If $$(x+y+z)P=xA+yB+zC, \text{ then}$$
$$\left(\frac{a^2}{x}+\frac{b^2}{y}+\frac{c^2}{z}\right)P''=\frac{a^2}{x}A+\frac{b^2}{x}B+\frac{c^2}{x}C.$$
The point P″ may be defined the *angular reciprocal* of P, then P is the angular reciprocal of P″.

For instance, the common intersection of AH, BK, CL (see 36), is the angular reciprocal of G the centre of gravity of ABC, therefore the angles BAG, CAH are equal.

Again, P, the common intersection of the perpendiculars of ABC, is the angular reciprocal of Q, the centre of the circumscribed circle; therefore the angles BAQ, CAP are equal.

[18]. By means of examples [16] and [17], we can,

by a geometrical construction, find the position of the point $a^nA+b^nB+c^nC$, where n is any integer, positive or negative. For instance, $a^3A+b^3B+c^3C$ is the angular reciprocal of the reciprocal of the centre of the inscribed circle.

CHAPTER II.

EQUATION OF FORCES.—MULTIPLICATION BY P.

120. NOTATION. Let the equation

$$p\mathrm{PP}' = a\mathrm{AA}' + b\mathrm{BB}' + c\mathrm{CC}' + \dots$$

mean that a force whose line of action is PP′, and which tends from P towards P′, and whose magnitude is p times PP′, is the resultant of the following forces: a force a times AA′ acting in the line AA′, and tending from A towards A′; a force b times BB′ acting in the line BB′, and tending from B toward B′; and so on.

121. *Definition. The above equation may be called an equation of forces.*

122. If two systems of forces be equal, then if we add to or subtract from them the same or equal systems of forces, the two resulting systems of forces will be equal to one another.

Again, if we diminish or increase each force of two equal systems of forces in the same ratio, the two resulting systems will be equal to one another.

Hence *the ordinary rules and processes in the solution of simple algebraic equations hold true with regard to equations of forces.*

123. *If* $(a+b+c+\ldots)G=aA+bB+cC+\ldots$
and P *be any point whatever, then*

$$(a+b+c+\ldots)PG \doteq aPA+bPB+cPC+\ldots$$

is a true equation of forces; that is, a force whose magnitude is $(a+b+c+\ldots)PG$, *acting in the line* PG *from* P *towards* G, *is the resultant of the following forces:* aPA *acting along* PA *from* P *towards* A, bPB *acting along* PB *from* P *towards* B; *and so on.*

Proof when A, B, C, etc., G, P are all in one plane.

Through P draw PX, PY, any two lines at right angles, and in the plane containing all the points. Let R, acting along the line PR, be the resultant of the forces aPA, bPB, etc.

Resolve parallel to PX, then

$$R\cos RPX=aPA\cos APX+bPB\cos BPX+\text{etc.}$$

But from $(a+b+c+\text{etc.})G=aA+bB+cC+\text{etc.}$ taking distances from PY, we get

$$(a+b+c+\text{etc.})PG\cos GPX$$
$$=aPA\cos APX+bPB\cos BPX+\text{etc.}$$

Hence $R\cos RPX=(a+b+c+\text{etc.})PG\cos GPX$ (1)

Similarly, resolving parallel to PY, we get

$$R\sin RPX=(a+b+c+\text{etc.})PG\sin GPX \qquad (2)$$

Squaring and adding the equations (1) and (2), and then taking square root, we get

$$R=(a+b+c+\text{etc.})PG \qquad \text{Q.E.D.}$$

Then from (1) and (2)

$$\cos RPX=\cos GPX,\ \sin RPX=\sin GPX$$

whence PR and PG must coincide. Q.E.D.

124. Reversing each force, it is now obvious that if

$$(a+b+c+\text{etc.})G=aA+bB+cC+\text{etc.},$$

then $(a+b+c+\text{etc.})GP=aAP+bBP+cCP+\text{etc.}$ is a true equation of forces. See 120.

125. *Multiplication by* P. Hence
when $(a+b+c+\ldots)G=aA+bB+cC+\ldots$ (1)
we obtain a true equation of forces by multiplying each term by P, only P must be put before each large letter G, A, B, C ..., or after each of them.

126. *Multiplication by* pP. Evidently from (1) we derive the following true equations of forces :

$$(a+b+c+\ldots)pPG=apPA+bpPB+cpPC+\ldots$$

and $(a+b+c+\ldots)pGP=apAP+bpBP+cpCP+\ldots$

127. *Multiplication by* $pP+qQ+rR+\ldots$
Let $(a+b+c+\ldots)G=aA+bB+cC+\ldots$
Then $p(a+b+c+\ldots)PG=apPA+bpPB+cpPC+\ldots$
is a true equation of forces.
Similarly

$$q(a+b+c+\ldots)QG=aqQA+bqQB+cqQC+\ldots$$

and $r(a+b+c+\ldots)RG=arRA+brRB+crRC+\ldots$
and so on ; therefore, adding

$$(a+b+c+\text{etc.})\,(pP+qQ+rR+\text{etc.})G$$
$$=(pP+qQ+rR+\text{etc.})\,(aA+bB+cC+\text{etc.})$$

where $(pP+qQ)\,(aA+bB)$ would mean the system $paPA+pbPB+qaQA+qbQB$ of forces.

128. *If* $(a+b+c+\ldots)G=aA+bB+cC+\ldots$ (1)
and $(x+y+z+\ldots)P=xA+yB+zC+\ldots$ (2)
then $(a+b+c+\ldots)(x+y+z+\ldots)GP$

$$=(aA+bB+cC+\ldots)\,(xA+yB+zC+\ldots)$$

is a true equation of forces.

For from (1) we derive as a true equation of forces

$$(a+b+c+\text{etc.})G\,(xA+yB+zC+\text{etc.})$$
$$=(aA+bB+cC+\text{etc.})\,(xA+yB+zC+\text{etc.}) \qquad (3)$$

and from (2) we derive as a true equation of forces

$$(x+y+z+\ldots)GP=G(xA+yB+zC+\ldots) \qquad (4)$$

hence from (3) and (4)

$$(a+b+c+\dots)(x+y+z+\dots)GP$$
$$=(aA+bB+cC+\dots)(xA+yB+zC+\dots)$$

If $(a+b+c+\dots)G=aA+bB+cC+\dots$

and $(x+y+z+\dots)P=xA+yB+zC+\dots$

then, multiplying, we get for a true equation of forces

$$(a+b+c+\dots)\ (x+y+z+\dots)\ GP$$
$$=(aA+bB+cC+\dots)\ (xA+yB+zC+\dots)$$

the terms cC, *x*A, *by multiplication, produce the force cx*CA, *and not the force cx*AC; *also a*A, *x*A, *by multiplication, produce ax*AA=*zero.*

129. Conversely: *If we can split an expression of the form ab*AB+*ac*AC+ ... +*bc*BC+*bd*BD+ ... +*cd*CD+ .. *into two factors of the form*

$$(xA+yB+zC+\dots)(pA+qB+rC+\dots)$$

and if $(x+y+z+\dots)P=xA+yB+zC+\dots$

and $(p+q+r+\dots)Q=pA+qB+rC+\dots$

then $(x+y+z+\dots)(p+q+r+\dots)PQ$ *is the resultant of the forces ab*AB, *ac*AC, ..., *bc*BC, ..., *cd*CD...

Examples [1]. Parallelogram of forces.

Let AB, AC be two forces. Complete the parallelogram ABDC; then 0=A—B+D—C. See 90.

and $AB+AC=A(B+C)=A(B+C-A)=AD$

therefore the forces represented by AB,AC have a resultant represented by AD. See 129.

[2]. Resultant of the forces pAB, qAC.

Divide BC in E, so that $pBE=qCE$,

then $(p+q)E=pB+qC$

since $pAB+qAC=A(pB+qC)=\overline{p+q}AE$

AE is the line of action of the resultant, and $\overline{p+q}AE$ is its magnitude.

[3]. To find the resultant of the forces PA, PB, PC.

Here $PA+PB+PC=P(A+B+C)=3PG$, where G is the CG of the triangle ABC. The resultant of PA, PB, PC has PG for line of action, and its magnitude is 3PG.

[4]. D, E, F being the middle points of the sides of the triangle ABC, the systems of forces PA, PB, PC and PD, PE, PF are equivalent.

For $PD+PE+PF=P(D+E+F)=\frac{1}{2}P(2D+2E+2F)$
$=\frac{1}{2}P(B+C+C+A+A+B)=P(A+B+C)=$
$PA+PB+PC$

[5]. To find the resultant of the forces pBC, qCA, rAB.

$$pBC+qCA+rAB=(pB-qA)C+rAB$$
$$=(p-q)DC+rAB.$$

where D is the point $(p-q)D=pB-qA$, and lies on AB. Hence the required resultant, being the resultant of two forces $(p-q)DC$ and rAB both through D, passes through D. Again,

$$pBC+qCA+rAB=qCA+B(pC-rA)$$
$$=qCA+(p-r)BE$$

where $(p-r)E=pC-rA$, and therefore E lies on CA. Hence the required resultant passes through E, therefore it acts in the line DE. Again,

$$(p-q)(p-r)DE=(pB-qA)(pC-rA)$$
$$=p^2BC-prBA-qpAC=p^2BC+prAB+pqCA.$$

hence $\dfrac{(p-q)(p-r)DE}{p}$ represents completely the required resultant.

[6]. What forces along the sides of the triangle ABC have a resultant in the line joining G, the centre

of gravity of the triangle with Q, the centre of its circumscribed circle?

Here $$3G = A + B + C$$

$$4 \sin A \sin B \sin C . Q = \sin 2A . A + \sin 2B . B + \sin 2C . C$$

multiplying, we get as a true equation of forces,

$$\begin{aligned} 12 \sin A \sin B \sin C . GQ &= \sin 2B . AB + \sin 2C . AC + \sin 2A . BA \\ &\quad + \sin 2C . BC + \sin 2A . CA + \sin 2B . CB \\ &= (\sin 2C - \sin 2B)\, BC + (\sin 2A - \sin 2C) CA \\ &\quad + (\sin 2B - \sin 2A) AB. \end{aligned}$$

Hence the required forces along BC, CA, AB, are proportional to

$$(\sin 2B - \sin 2C) \sin A,\ (\sin 2C - \sin 2A) \sin B,\ (\sin 2A - \sin 2B) \sin C.$$

[7]. To find what forces along the sides and diagonals of a quadrilateral ABCD have a resultant represented by the line joining the bisections E,F of AB,CD.

Since $$2E = A + B \text{ and } 2F = C + D$$

hence $4EF = (A+B)(C+D)$ is a true equation of forces,

therefore $$EF = \tfrac{1}{4}AC + \tfrac{1}{4}AD + \tfrac{1}{4}BC + \tfrac{1}{4}BD.$$

The required forces are represented in magnitude and direction by $\frac{1}{4}AC$, $\frac{1}{4}AD$, $\frac{1}{4}BC$, $\frac{1}{4}BD$.

[8]. If ABCD be a quadrilateral; and α, β, γ, δ, denote the areas of the triangles BCD, CDA, DAB, ABC; then $0 = \alpha\beta AB - \beta\gamma BC + \gamma\delta CD - \delta\alpha DA$ is a true equation of forces; that is, forces represented by $\alpha\beta AB$, $\gamma\delta CD$ are balanced by forces $\beta\gamma CB$, $\delta\alpha AD$.

For we have $$0 = \alpha A - \beta B + \gamma C - \delta D \qquad (1)$$

$$\times A$$

therefore $$0 = -\beta BA + \gamma CA - \delta DA \qquad (2)$$

Again multiply (1) by C

then $\qquad 0=\alpha AC-\beta BC-\delta DC \qquad (3)$

$\alpha(2)+\gamma(3)$ give, after reductions, the required result.

Cor. 1. When ABCD is a parallelogram

$$0=A-B+C-D$$

hence $0=AB-BC+CD-DA$ is a true equation of forces.

Cor. 2. When ABCD is a trapezium of which the parallel sides are BC, AD

$$0=BC.A-AD.B+AD.C-BC.D$$

hence $0=BC.AD.AB-AD^2.BC+AD.BC.CD-BC^2.DA$ is a true equation of forces, the forces acting along AB, BC, CD, DA.

Reducing we get, that a force AB along AB, a force AD along CB, a force CD along CD, and a force BC along AD, form a system in equilibrium.

Cor. 3. ABCD being a quadrilateral inscribed in a circle

$$\begin{aligned}0=&BC.CD.DB.A-CD.DA.AC.B\\&+DA.AB.BD.C-AB.BC.CA.D\end{aligned}$$

from which we derive the true equation of forces

$$\begin{aligned}0=&BC.CD.DB.CD.AD.AC.AB\\&-CD.DA.AC.DA.AB.BD.BC\\&+DA.AB.BD.AB.BC.CA.CD\\&-AB.BC.CA.BC.CD.DB.DA\end{aligned}$$

AB, BC, CD, DA being the lines of action of the forces: reducing, we infer, that a force CD acting along AB, a force DA along CB, a force AB acting along CD, and a force BC along AD, form a system in equilibrium.

This may be expressed by the equation of forces

$$0=\frac{CD}{AB}.AB-\frac{DA}{BC}.BC+\frac{AB}{CD}.CD-\frac{BC}{DA}.DA.$$

[9]. The middle points XYZ of the diagonals of complete quadrilateral ABCD lie in one straight line. See figure of article 79.

$$4XY+4YZ=(A+C)\,(B+D)+(B+D)\,(F+G)$$
$$=AB+AD+CB+CD+BF+BG+DF+DG$$
$$=(AB+BF)+(AD+DG)+(CB+BG)+(CD+DF)$$
$$=AF+AG+CG+CF$$
$$=A(F+G)+C(F+G)$$
$$=(A+C)\,(F+G)$$
$$=4XZ$$

or $XY+YZ=XZ$

whence X, Y, Z must be in one right line.

[10]. If a polygon ABCD... circumscribes a circle, and P, Q, R... be the points of contact of the sides AB, BC, CD,..., and O the centre of the circle, then forces proportional to AB, BC, CD,... acting along OP, OQ, OR,..., are in equilibrium.

Since $(AB+BC+CD+\ldots)O$

$$=AB.P+BC.Q+CD.R+\ldots \qquad \text{See 118.}$$

Multiply by O

hence $\quad 0=AB.OP+BC.OQ+CD.OR+\text{etc.}$

is a true equation of forces, these forces acting along OP, OQ, OR,... and being proportional to AB.OP, BC.OQ, etc., that is to AB, BC, etc., since OP, OQ, etc., are equal radii.

CHAPTER III.

POINTS IN ONE STRAIGHT LINE.

130. *If* A, B, C,...*be points in one straight line, and if*

$$(a+b+c+\ldots)G=aA+bB+cC+\ \ldots \qquad (1)$$

and P *be any point in that line, then*

$$(a+b+c+\ldots)PG=aPA+bPB+cPC+\ldots$$

where PG *being positive,* PA *will also be positive, if* PG, PA *point towards the same direction, but* PA *will be negative if* PG, PA *point in opposite directions, and likewise with* PB, PC...

Multiplying (1) by P we get the following true equation of forces, see 123,

$$(a+b+c+\ldots)PG=aPA+bPB+cPC+\ldots\ldots\ldots \quad (2)$$

Now forces, in one right line, having for resultant their algebraic sum, it follows that (the convention above described being taken into consideration) (2) may be considered a true equation of distances.

131. Similarly:

If $\qquad (x+y+z+\ldots)P=xX+yY+zZ+\ldots$

and $\qquad (a+b+c+\ldots)G=aA+bB+cC+\ldots$

where A,B,C...; X,Y,Z...*all lie in one right line; we get as a true geometrical equation* $(a+b+c+\ldots)(x+y+z+\ldots)GP$

$$=(aA+bB+cC+\ldots)\ (xX+yY+zZ+\ldots)$$

where the term axAX *produced by multiplication is posi-*

tive when AX *points in the same direction as* GP, *but negative when pointing in opposite direction.*

Examples. [1]. If a line AB be bisected in C, and produced to P, then $2PC=PA+PB$.
For $2C=A+B$, multiply by P, therefore $2PC=PA+PB$.

[2]. If a line AB be bisected in C, and unequally divided in P, then $2PC=PA-PB$.

Take the case in which P lies between C and B, since $2C=A+B$, multiply by P, therefore $2PC=PA+(-PB)$.

[3]. In the triangle ABC, if D be the middle point of BC and E the foot of the bisector of the interior angle A, then $DE=\frac{1}{2}\frac{b-c}{b+c}a$.

Take the case in which D lies on EC
since $(b+c)E=bB+cC$
multiply by D
hence $(b+c)DE=bDB+c(-DC)=b\frac{a}{2}-c\frac{a}{2}$, hence, etc.

[4]. To find the distance between the feet E, F of the bisectors of the interior and exterior angles at A of the triangles ABC.

Take the case in which F lies on CB produced
since $(b+c)E=bB+cC$ and $(b-c)F=bB-cC$,
therefore $(b^2-c^2)EF=-bcBC+bcCB=2bcCB$
is a true equation of forces. These forces being in one right line, we get, for a true geometrical equation

$$(b^2-c^2)EF=2bcCB=2abc$$

which gives the magnitude of EF.

[5]. If D be the middle point of BC,E the foot of the perpendicular from A on BC, q the radius of the circle circumscribing ABC, then

$$DE=q \sin (B—C)$$

[6]. If ABCD be points in order in a straight line, then AC.BD=AB . CD+AD .BC. (Townshend, p. 102.)

For $\quad AC . B=AB . C+BC . A$

multiply by D, therefore AC . DB=AB . DC+BC . DA.

Corollary. Hence, if ABCD be a line harmonically divided, AC . BD=2AB . CD=2BC . DA, for then BC . DA=AB . DC.

[7]. If O be the mean point of n points A, B, C... all in one straight line, and if O′ be the mean of n other points A′ B′ C′... also in the same line, then

$$nOO'=AA'+BB'+CC'+\dots \text{ (Townshend, p. 96.)}$$

For $\quad nO \ =A+B+C+\dots \qquad (1)$

and $\quad nO' \ =A'+B'+C'+\dots \qquad (2)$

Let P be any point in the line, leaving all the points A, B, C...and therefore O also, and A′ B′ C′... and therefore O′ on the same side.

Then (1) P gives $nPO =PA+PB+PC+\dots$

and (2) P gives $\quad nPO'=PA'+PB'+PC'+\dots$

subtract $n(PO—PO')=(PA—PA')+(PB—PB')+\dots$

or $\qquad nOO'=AA'+BB'+\text{etc.}$

Of course some of the terms may be negative.

[8]. If O be the mean of n points ABC...all in one straight line, and O′ the mean of n' points A′ B′ C′... in the same line,

then $\qquad nn' OO'=AA'+AB'+AC'+\dots$
$\qquad\qquad +BA'+BB'+BC'+\dots+CA'+\text{etc.}$

For $\qquad nO =A+B+C+\dots$

and $\qquad nO'=A'+B'+C'+\dots$

multiplying we get for a true equation of forces

$$nn'\,OO'=AA'+AB'+AC'+\ldots+BA'+BB'+BC'+\ldots$$
$$+CA'+\ldots$$

now OO′, AA′, AB′,...being in one right line, this equation of forces is readily turned into a true geometrical equation, by giving the sign + to OO′, and the sign — to any one of the lines AA′, AB′, ..., BA′... which points in a direction opposite to that of OO′.

CHAPTER IV.

DISTANCES FROM A STRAIGHT LINE.

132. *If* $(a+b+c+\text{etc.})G=aA+bB+cC+\text{etc.}$ (1)
and g', a', b', c', *etc., be perpendiculars from* G, A, B, C, *etc., upon the same straight line* LM, *or be parallel lines drawn from these points to meet the same straight line* LM, *then*

$$(a+b+c+\text{etc.})g'=aa'+bb'+cc'+\text{etc.}; \qquad (2)$$

to the perpendiculars which are on one side of LM *the same sign must be given, to those on the other side of* LM *the opposite sign must be given.*

(1) means that G is the centre of parallel forces, a at A, b at B, c at C, etc.; therefore taking distances from any straight line LM, either perpendicular to it or parallel to any other straight line, we derive the truth of equation (2). See article 3.

Examples. [1]. The distance of G, the centre of gravity of ABC, from one side BC is one-third of the distance of the opposite vertex A from the same side.

Draw AD, GE perpendiculars on BC,

since $$3G=A+B+C$$

taking distances from BC, therefore $3GE=AD$.

[2]. To find a point in the base BC of a triangle ABC, such that if perpendiculars be drawn from it

upon the sides, their sum shall be equal to a given line σ.

Let β, γ be the perpendiculars from B, C on the opposite sides, and let $(x+y)\text{P}=x\text{B}+y\text{C}$ (1) be the required point. The position of P will be known if we find the ratio of x to y.

Let PQ, PR be perpendiculars from P on BA, CA in (1), take distances from CA and BA;

then $\quad (x+y)\text{PR}=x\beta$

and $\quad (x+y)\text{PQ}=y\gamma$

add $\quad (x+y)\sigma=x\beta+y\gamma$ (see question)

hence $\quad x:y=\gamma-\sigma:\sigma-\beta$

and therefore the required point is, see (1)

$$(\gamma-\sigma)\text{B}+(\sigma-\beta)\text{C}$$

[3]. If lines AL, AK be drawn making equal angles with the bisector AD of the side BC of the triangle ABC, the sums of the perpendiculars on them from B and C are equal.

Let DK, BK′, CK″ be perpendiculars on AK.

Let DL, BL′, CL″ be perpendiculars on AL.

Then in the case where AK, AL lie in the angles BAD, DAC;

since $\quad 2\text{D}=\text{B}+\text{C}$, taking distances from AK

$$2\text{DK}=-\text{BK}'+\text{CK}''$$

similarly $\quad 2\text{DL}=\text{BL}'-\text{CL}''$

but since the angle KAL is bisected by AD,

therefore $\quad \text{DK}=\text{DL}$

hence $\quad -\text{BK}'+\text{CK}''=\text{BL}'-\text{CL}''$

therefore $\quad \text{BK}'+\text{BL}'=\text{CK}''+\text{CL}''$.

[4]. Let p, p', p'' denote the perpendiculars of the triangle ABC; let q, q', q'' denote the perpendiculars drawn from any point Q on BC, CA, AB, then

$$1=\frac{q}{p}+\frac{q'}{p'}+\frac{q''}{p''}$$

For let $(x+y+z)Q=xA+yB+zC$

take distances from BC, CA, AB successively

therefore $(x+y+z)q=xp$

$(x+y+z)q'=yp'$

$(x+y+z)q''=zp''$

whence $\frac{q}{p}+\frac{q'}{p'}+\frac{q''}{p''}=1.$

[5.] P is any point within the triangle ABC; DE, FG, HK are lines through P parallel to BC, CA, AB, and such that K, G lie on BC; D, H on CA; F, E on AB. To prove PD.PF.PK=PH.PE.PG

Let $(x+y+z)P=xA+yB+zC.$

Take distances from CA parallel to BC, then

$$(x+y+z)PD=y.BC \qquad (1)$$

Take distances from AB parallel to BC, then

$$(x+y+z)PE=zBC \qquad (2)$$

similarly $(x+y+z)PF=zCA \qquad (3)$

and $(x+z+z)PG=xCA \qquad (4)$

also $(x+y+z)PK=xAB \qquad (5)$

$$(x+y+z)PH=yAB \qquad (6)$$

from these six equations PD.PF.PK=PE.PG.PH.

[6.] If DEF be a line meeting BC, CA, AB in D, E, F, and if AA′, BB′, CC′ be drawn through A, B, C, parallel to DE, to meet BC, CA, AB in A′, B′, C′, and if G be any point on DE, then

$$\frac{GD}{AA'}+\frac{GE}{BB'}+\frac{GF}{CC'}=1$$

For let $(a+b+c)G=aA+bB+cC$

Take distances from BC parallel to DE, then

$$(a+b+c)GD=a.AA'$$

similarly $(a+b+c)\mathrm{GE}=b\mathrm{BB}'$

and $(a+b+c)\mathrm{GF}=c\mathrm{CC}'$

hence $$\frac{\mathrm{GD}}{\mathrm{AA}'}+\frac{\mathrm{GE}}{\mathrm{BB}'}+\frac{\mathrm{GF}}{\mathrm{CC}'}=1.$$

[7.] If any three equal lines, AD, BE, CF, be drawn from the angles of a triangle ABC to the opposite sides or these produced, and from any point O within it lines, OD′, OE′, OF′, parallel to these be drawn to the sides, the sum of these latter lines shall be equal to either of the former.

Let $(x+y+z)\mathrm{O}=x\mathrm{A}+y\mathrm{B}+z\mathrm{C}$

take distances from BC parallel to AD, then

$$(x+y+z)\mathrm{OD}'=x\mathrm{AD}$$

similarly $(x+y+z)\mathrm{OE}'=y\mathrm{BE}=y\mathrm{AD}$

and $(x+y+z)\mathrm{OF}'=z\mathrm{CD}=z\mathrm{AD}$

adding and reducing $\mathrm{OD}'+\mathrm{OE}'+\mathrm{OF}'=\mathrm{AD}$.

133. [8]. *If* $(x+y+z)\mathrm{P}=x\mathrm{A}+y\mathrm{B}+z\mathrm{C}$, *and perpendiculars* PD, PE, PF *be drawn on* BC, CA, AB, *the area of the triangle* DEF *is*

$$\tfrac{1}{2}\frac{(2\,\text{area ABC})^3}{a^2b^2c^2(x+y+z)^2}(yza^2+zxb^2+xyc^2).$$

For area DEF = area PEF + area PFD + area PDE

but area PEF $=\frac{1}{2}$PE . PF sin A

in $(x+y+z)\mathrm{P}=x\mathrm{A}+y\mathrm{B}+z\mathrm{C}$, take distances from CA

then $$(x+y+z)\mathrm{PE}=y\,\frac{2\,\text{area ABC}}{b}$$

similarly $$(x+y+z)\mathrm{PF}=z\,\frac{2\,\text{area ABC}}{c}$$

also $$\sin\mathrm{A}=\frac{2\,\text{area ABC}}{bc}$$

therefore $$\text{area PEF}=\tfrac{1}{2}\,\frac{(2\,\text{area ABC})^3}{a^2b^2c^2(x+y+z)^2}yza^2\,;$$

and similar expressions are obtained for PFD, PDE; adding, we get the result required.

[9]. If from the angles C, D of a quadrilateral ABCD, and from E, the intersection of its diagonals, perpendiculars CF, DG, EH be dropped upon AB, its area $= \frac{1}{2}\frac{AB \,.\, CF \,.\, DG}{EH}$

For $0 = BCD \,.\, A - CDA \,.\, B + DAB \,.\, C - ABC \,.\, D$

whence, area quadrilateral $.\, E = BCD \,.\, A + DAB \,.\, C$

Take distances from AB, therefore

area quadrilateral $.\, EH = DAB \,.\, CF = \frac{1}{2} AB \,.\, DG \,.\, CF$.

[10]. In a trapezium ABCD, the line EF drawn through the middle point E of one AB of the non-parallel sides, and terminated at the other, is half the sum of the parallel sides.

For $2E = A + B$

Take distances from CD parallel to BC,

then $2EF = AD + BC$; therefore, etc.

[11]. ABCD is a parallelogram; EF any line without it; to prove that the sum of the perpendiculars AA', CC' from the opposite vertices A,C on EF, is equal to the sum of the perpendiculars BB', DD' on the same line.

For $0 = A - B + C - D$ See 90.

Take distances from EF; therefore

$0 = AA' - BB' + CC' - DD'$; hence, etc.

[12]. The perpendicular from the internal centre of similitude of two circles upon their external common tangent, is the harmonic mean between the radii.

Let A, B be the centres of the circles; let a, b be their radii; let one external common tangent touch the

circles A, B in P, Q; let C be the internal centre of similitude; draw CC′ perpendicular on PQ.

$$\left(\frac{1}{a}+\frac{1}{b}\right)\mathrm{C}=\frac{\mathrm{A}}{a}+\frac{\mathrm{B}}{b} \qquad \text{See 113.}$$

Take distances from PQ; therefore

$$\left(\frac{1}{a}+\frac{1}{b}\right)\mathrm{CC'}=\frac{\mathrm{AP}}{a}+\frac{\mathrm{BQ}}{b}=1+1=2$$

therefore $$\frac{2}{\mathrm{CC'}}=\frac{1}{a}+\frac{1}{b} \qquad \text{Q.E.D.}$$

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134. [13]. *If* $(a+b+c+\ldots)\mathrm{G}=a\mathrm{A}+b\mathrm{B}+c\mathrm{C}+\ldots$ *and with centre* G *and any radius* r *we describe a circle, and draw any tangent to it, then* $a', b', c', \ldots$ *being the perpendiculars on that tangent from* A, B, C, ..., *we have*

$$aa'+bb'+cc'+\ldots \quad =\textit{constant}=(a+b+c+\ldots)r.$$

This result is obtained at once by taking distances from the tangent.

Corollary. $aa'+bb'+cc'+\text{etc.}=0$ for any line drawn through G. This is also an immediate consequence of Article 132.

[14]. The sum of the perpendiculars AA′, BB′, CC′, from the vertices of the equilateral triangle ABC upon any tangent to its circumscribing circle is three times the radius q of that circle.

For, let Q be the centre of the circle, since ABC is equiangular,

$$3\mathrm{Q}=\mathrm{A}+\mathrm{B}+\mathrm{C}$$

Therefore, taking distances from the tangent,

$$3q=\mathrm{AA'}+\mathrm{BB'}+\mathrm{CC'}.$$

[15]. The *algebraic* sum of the perpendiculars from the vertices of the same equilateral triangle ABC upon

any tangent to the inscribed circle, is readily proved to be half the preceding sum.

[16]. If AA′, BB′, CC′ be perpendiculars from the vertices of any triangle ABC, upon any tangent of the inscribed circle, then

$$a\mathrm{AA}'+b\mathrm{BB}'+c\mathrm{CC}'=\text{twice the area of ABC},$$

the convention of signs being attended to.

Let O be the centre of the inscribed circle, r its radius, then $$(a+b+c)\mathrm{O}=a\mathrm{A}+b\mathrm{B}+c\mathrm{C}.$$

Take distances, for instance, from a tangent which has A on one side and B, C on the other side of itself, then

$$(a+b+c)r=a(-\mathrm{AA}')+b\mathrm{BB}'+c\mathrm{CC}'$$

but $(a+b+c)r=$twice the area of the triangle ABC.

[17]. Since $4\mathrm{Q}=\mathrm{O}+\mathrm{O}_1+\mathrm{O}_2+\mathrm{O}_3$. See 54.

The algebraic sum of the perpendiculars from the centres of the inscribed and escribed circles, upon any tangent to the circumscribed circle of the triangle ABC, is double of the diameter of the circumscribed circle.

[18]. If ABCD... be a regular polygon of n sides circumscribing a circle, radius r, centre O; and any tangent be drawn to the circle; then the sum of those perpendiculars on it from ABCD..., which lie on the same side of the tangent as O does, exceeds by nr the sum of those lying on the other side.

Since $$n\mathrm{O}=\mathrm{A}+\mathrm{B}+\mathrm{C}+\dots$$

Take distances from the tangent; hence, etc.

[19]. The two sums of the perpendiculars from the alternate angles of a regular polygon of an even number of sides, upon any tangent to its circumscribing circle, are equal.

CHAPTER V.

PROJECTION ON A RIGHT LINE OF A SYSTEM OF POINTS.

135. *If* $(a+b+c+\ldots)G=aA+bB+cC+\ldots$ (1) *and* G′, A′, B′, C′,... *be the orthogonal projections of* G, A, B, C,... *on any straight line, then*

$$(a+b+c+\ldots)G'=aA'+bB'+cC'+\ldots$$

For produce GG′, AA′, BB′, CC′,...by lengths equal to their own, thus obtaining a system of points

$$G'', A'', B'', C'',\ldots$$

perfectly superposable upon the system GABC...

Hence $(a+b+c+\ldots)G''=aA''+bB''+cC''+\ldots$ (2) add (1) and (2) and take half, and remark that

$$2G'=G+G'',\ 2A'=A+A'', \text{ etc.}$$

therefore $(a+b+c+\ldots)G'=aA'+bB'+cC'+\ldots$

Examples. [1]. In any triangle ABC, if BE, CF be perpendiculars on any line through A, and D be the bisection of BC, show that DE=DF.

Draw DM perpendicular on AE

we have $2D=B+C$

project on the line AE

$$2M=E+F$$

that is, EM=FM, but DM is common to the two right-angled triangles EDM, DMF. These equal triangles give DE=DF.

[2]. ABC is an equilateral triangle, G its CG; S is any point whose projections on BC, CA, AB are P, Q, R, to prove that G′, the centre of gravity of the triangle PQR, bisects SG.

Let $$(x+y+z)S=xA+yB+zC$$
project on BC, therefore, remarking that the projection of A is the middle point of BC,

$$(x+y+z)P=x\frac{B+C}{2}+yB+zC$$

or $$2(x+y+z)P=(x+2y)B+(x+2z)C$$
similarly $$2(x+y+z)Q=(y+2z)C+(y+2x)A$$
and $$2(x+y+z)R=(z+2x)A+(z+2y)B$$
adding the last three equations we get

$$2(x+y+z)(P+Q+R)$$
$$=(x+y+z)(A+B+C)+3(xA+yB+zC)$$
or $$6(x+y+z)G'=3(x+y+z)G+3(x+y+z)S$$
or $$2G'=G+S \qquad \text{Q.E.D.}$$

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136. *If* $(a+b+c+\ldots)G=aA+bB+cC+\ldots \quad (1)$ *and* G′, A′, B′, C′,... *be the projections of* G, A, B, C... *upon any right line, these projections being made parallel to any other right line, then will*

$$(a+b+c+\ldots)G'=aA'+bB'+cC'+\ldots$$

In (1) take distances from the first line parallel to the second, then

$$(a+b+c+\ldots)GG'=aAA'+bBB'+cCC'+\ldots \qquad (2)$$

but AA′, GG′ being parallel

$$\frac{A-A'}{AA'}=\frac{G-G'}{GG'} \qquad \text{See 84.}$$

therefore $$A=A'+\frac{AA'}{GG'}G-\frac{AA'}{GG'}G'$$

with similar expressions for B C, ...

Substituting for A, B, C, ... in (1) we get

$$(a+b+c+\text{etc.})G = aA'+bB'+cC'+\text{etc.}$$
$$+(aAA'+bBB'+cCC'+\text{etc.})\frac{G}{GG'}$$
$$-(aAA'+bBB'+\text{etc.})\frac{G'}{GG'}$$

hence, using equation (2)

$$(a+b+c+\text{etc.})G = aA'+bB'+cC'+\text{etc.}$$
$$+(a+b+c+\text{etc.})G-(a+b+c+\text{etc.})G'$$

whence the required result.

Example. If three lines, MN, PQ, RS, be drawn through any point, Z, within a triangle parallel to the sides AC, AB, BC, then will

$$\frac{MN}{AC}+\frac{PQ}{AB}+\frac{RS}{BC}=2 \quad \text{(Morgan's Problems, p. 8.)}$$

Draw a figure in which M, R lie on AB; Q, N on BC; and S, P on CA.

Let $(x+y+z)Z = xA+yB+zC$

project on AB, AC parallel to BC, then

$$(x+y+z)R = xA+(y+z)B$$
$$(x+y+z)S = xA+(y+z)C$$

Hence $(x+y+z)(R-S) = (y+z)(B-C)$

therefore $$\frac{RS}{BC}=\frac{y+z}{x+y+z}$$ See 85.

similarly $$\frac{MN}{CA}=\frac{z+x}{x+y+z},\quad \frac{PQ}{AB}=\frac{x+y}{x+y+z}$$

by adding and reducing the required result is obtained.

CHAPTER VI.

AREAS.—MULTIPLICATION BY PQ.

137. *Multiplication by* PQ.

If $(a+b+c+\ldots)G = aA + bB + cC + \ldots$ (1)
and P, Q *be any two points, then*
$(a+b+c+\ldots)$ area $GPQ = a$ area $APQ + b$ area $BPQ + \ldots$
opposite signs being given to triangles having the common base PQ *and vertices on opposite sides of it.*

Let GG′, AA′, BB′, CC′, ... be the perpendiculars from G, A, B, C...on PQ; in (1) take distances from PQ, therefore

$$(a+b+c+\ldots)GG' = aAA' + bBB' + cCC' + \ldots$$

multiply this true geometrical equation by $\frac{1}{2}$PQ, therefore

$$(a+b+c+\ldots) \text{ area } GPQ = aAPQ + bBPQ + cCPQ + \ldots$$

the triangles having the same sign when the perpendiculars have the same sign, that is, when their vertices are on the same side of the common base PQ.

Examples. [1]. If C bisects AB, and the line PQ does not intersect AB, then the triangle PCQ is the semi-sum of the triangles PAQ, PBQ.

For $2C = A + B$
multiply by PQ
therefore $2CPQ = APQ + BPQ$.

[2]. G being the centre of gravity of the triangle ABC, the triangles GBC, GCA, GAB, are equal.

For $\quad 3G=A+B+C$

$\quad\quad \times BC$

therefore $\quad 3GBC=ABC$

similarly $\quad 3GCA=ABC=3GAB$, hence, etc.

[3]. If triangles AEF, ABC have a common angle A, they are in the ratio of AE . AF to AB . AC.

In your figure place C on AE, and B on AF

then $\quad AF.B=AB.F+BF.A \quad (1)$

and $\quad AE.C=AC.E+CE.A \quad (2)$

(1) × AE gives $\quad$ AF area BAE = AB area FAE

(2) × BA gives $\quad$ AE area CBA = AC area EBA

multiplying the last two equations, and reducing, the required result is obtained.

[4]. ABCD is a parallelogram, and E any point in the diagonal AC or AC produced; show that the triangles EBC, EDC are equal.

Here $\quad 0=A-B+C-D$

$\quad\quad \times EC$

therefore $\quad 0=0-$ area $BEC+0-(-DEC)$

hence, etc.

[5]. If from a point P, without a parallelogram ABCD, lines be drawn to the extremities of two adjacent sides, BA, BC, and of the diagonal BD which they include; of the triangles thus formed, that whose base is the diagonal is equal to the sum of the other two.

Proof, when P lies in the space included by AB, CB produced, DA produced.

Since $\quad 0=A-B+C-D$

$\quad\quad \times PB$

therefore $\quad 0=APB-0+CPB-DPB.$

[6]. The two triangles formed by drawing lines from any point P within a parallelogram ABCD, to the extremities of two opposite sides AB, CD, are together half the parallelogram.

Suppose P within the triangle ACD,

since $$\mathrm{D}=\mathrm{A}-\mathrm{B}+\mathrm{C}$$

$$\times\mathrm{PC}$$

therefore $$\mathrm{PCD}=-\mathrm{PCA}+\mathrm{PCB},$$ add PAB to each,

then $$\mathrm{PAB}+\mathrm{PCD}=\mathrm{PAB}+\mathrm{PCB}-\mathrm{PCA}=\mathrm{ABC}$$

=one half of the parallelogram.

[7]. ABCD, AECF are two parallelograms ; EA, AD lying in one straight line. Let FG be drawn parallel to AC, to meet BA produced in G, then the triangle ABE equals the triangle ADG.

Let E lie on DA produced; let $\mathrm{AD}=p=\mathrm{BC}$; let $\mathrm{AE}=q=\mathrm{FC}$,

then $$(p+q)\mathrm{C}=q\mathrm{B}+p\mathrm{F};$$

project on AB by lines parallel to AC See 136.

then $$(p+q)\mathrm{A}=q\mathrm{B}+p\mathrm{G},$$

but $$(p+q)\mathrm{A}=p\mathrm{E}+q\mathrm{D};$$

hence $$q\mathrm{B}+p\mathrm{G}=p\mathrm{E}+q\mathrm{D}.$$

Multiply by GE.

q area $\mathrm{BGE}=q$ area DGE, or $\mathrm{BGE}=\mathrm{DGE}$; take away the common triangle AGE, hence $\mathrm{AEB}=\mathrm{GAD}$.

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138. *If* $(x+y+z+\ldots)\mathrm{P}=x\mathrm{A}+y\mathrm{B}+z\mathrm{C}+\ldots$ (1)

and $(a+b+c+\ldots)\mathrm{Q}=a\mathrm{A}+b\mathrm{B}+c\mathrm{C}+\ldots$ (2)

and $(p+q+r+\ldots)\mathrm{R}=p\mathrm{A}+q\mathrm{B}+r\mathrm{C}+\ldots$ (3)

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then

$$(x+y+z+\text{etc.})(a+b+c+\text{etc.})(p+q+r+\text{etc.}) \text{ area PQR}$$
$$=(x\text{A}+y\text{B}+z\text{C}+\text{etc.})(a\text{A}+b\text{B}+\text{etc.})(p\text{A}+q\text{B}+\text{etc.})$$

where by multiplication xA, bB, rC *produce* xbr *area* ABC ; *and sign* + *being given to the triangle* PQR, *sign* + *will also be given to the triangle* ABC, *if travelling from* A *to* B, *and thence to* C, *we go round in the same direction as when moving from* P *to* Q, *and thence to* R.

From (1) and (2) we derive as a true equation of forces

$$(x+y+z+\ldots)(a+b+c+\ldots)\text{PQ}$$
$$=(x\text{A}+y\text{B}+\text{etc.})(a\text{A}+b\text{B}+\text{etc.})$$

Take half moments about R, therefore

$$(x+y+z+\text{etc.})(a+b+c+\text{etc.}) \text{ area PQR}$$
$$=(x\text{A}+y\text{B}+z\text{C}+\text{etc.})(a\text{A}+b\text{B}+c\text{C}+\text{etc.})\text{R}$$
$$=(xb\text{AB}+xc\text{AC}+\ldots+ya\text{BA}+\ldots)\text{R} \qquad (4)$$

Now, $\quad (p+q+r+\text{etc.})\text{R}=p\text{A}+q\text{B}+r\text{C}+\ldots$

therefore

$$(p+q+r+\text{etc.})\text{ABR}=\text{AB}(p\text{A}+q\text{B}+r\text{C}+\ldots)$$

therefore

$$xb(p+q+r+\text{etc.})\text{ABR}=xb\text{AB}(p\text{A}+q\text{B}+r\text{C}+\ldots)$$

similarly

$$xc(p+q+r+\text{etc.})\text{ACR}=xc\text{AC}(p\text{A}+q\text{B}+r\text{C}+\ldots)$$

and so on. Adding, we get

$$(p+q+r+\text{etc.})(xb\text{AB}+xc\text{AC}+\ldots)\text{R}$$
$$=(xb\text{AB}+xc\text{AC}+\ldots)(p\text{A}+q\text{B}+r\text{C}+\ldots)$$
$$=(x\text{A}+y\text{B}+\text{etc.})(a\text{A}+b\text{B}+\ldots)(p\text{A}+q\text{B}+\ldots)$$

(4) now gives

$$(x+y+z+\text{etc.})(a+b+c+\text{etc})(p+q+r+\text{etc.})\text{area PQR}$$
$$=(x\text{A}+y\text{B}+\text{etc.})(a\text{A}+b\text{B}+\text{etc.})(p\text{A}+q\text{B}+\text{etc.})$$

Examples. [1]. D, E, F being the bisections of BC, CA, AB, the area DEF is one-fourth of the area ABC.

Since $2D=B+C$, $2E=C+A$, $2F=A+B$, therefore

$$8 \text{ area } DEF=(B+C)(C+A)(A+B)$$
$$=(BC+BA+CA)(A+B)=BCA+CAB=2ABC.$$

[2]. To bisect a triangle ABC by a line DE parallel to BC.

Let $(x+y)\,D=xA+yB$, then $(x+y)\,E=xA+yC$,

hence $(x+y)^2DEA=(xA+yB)(xA+yC)A$
$$=(xyAC+xyBA+y^2BC)A=y^2ABC,$$

therefore $DEA:ABC=y^2:(x+y)^2$;

but by question $DEA:ABC=1:2$,

hence $y^2:(x+y)^2=1:2$ and $x:y=\sqrt{2}-1:1$.

And D the required point is $(\sqrt{2}-1)A+B$.

[3]. On the sides BC, CA, AB of the triangle ABC, the points D, E, F are taken such that $BD=DC$, $CE=2EA$, $AF=3FB$. To find the area of the triangle DEF.

Since $2D=B+C$, $3E=C+2A$, $4F=A+3B$,

therefore $2.3.4\,DEF=(B+C)(C+2A)(A+3B)$
$$=(BC+2BA+2CA)(A+3B)=BCA+6CAB=7ABC,$$

or $$DEF=\frac{7}{24}ABC.$$

[4]. To find the area of a triangle DEF whose angular points divide the sides of the triangle ABC in the ratio of n to 1.

Since

$(n+1)D=B+nC, (n+1)E=C+nA, (n+1)\,F=A+nB,$

therefore multiplying

$$(n+1)^3\mathrm{DEF}=(\mathrm{B}+n\mathrm{C})(\mathrm{C}+n\mathrm{A})(\mathrm{A}+n\mathrm{B})$$
$$=(\mathrm{BC}+n\mathrm{BA}+n^2\mathrm{CA})(\mathrm{A}+n\mathrm{B})$$
$$=\mathrm{BCA}+n^3\mathrm{CAB}=(n^3+1)\mathrm{ABC};$$

hence $$\mathrm{DEF}=\frac{n^2-n+1}{(n+1)^2}\mathrm{ABC}.$$

[5]. If D, E, F be the middle points of BC, CA, AB, and P any point within the triangle DEF, then

$$\text{area PBC}=2(\mathrm{PDE}+\mathrm{PDF}).$$

This result must be obtained from area PBC by expressing B and C in terms of D, E, F.

Now BDEF is a parallelogram, therefore

$$\mathrm{B}=\mathrm{D}-\mathrm{E}+\mathrm{F},$$

similarly $$\mathrm{C}=\mathrm{E}-\mathrm{F}+\mathrm{D},$$

therefore, $$\text{area PBC}=\mathrm{P}(\mathrm{D}-\mathrm{E}+\mathrm{F})(\mathrm{E}-\mathrm{F}+\mathrm{D})$$
$$=\mathrm{P}(\mathrm{DE}-\mathrm{DF}+\mathrm{EF}-\mathrm{ED}+\mathrm{FE}+\mathrm{FD})=\mathrm{P}(2\mathrm{DE}+2\mathrm{FD})$$

or $$\text{triangle PBC}=2\mathrm{PDE}+2\mathrm{PFD}$$

Corollary. Similarly $$\mathrm{PCA}=2\mathrm{PDE}+2\mathrm{PEF}$$

and $$\mathrm{PAB}=2\mathrm{PEF}+2\mathrm{PFD}$$

adding the squares $$(\mathrm{PBC})^2+(\mathrm{PCA})^2+(\mathrm{PAB})^2$$
$$=4\{(\mathrm{PEF})^2+(\mathrm{PFD})^2+(\mathrm{PDE})^2+(\mathrm{DEF})^2\}$$

hence $$4\{(\mathrm{PBC})^2+(\mathrm{PCA})^2+(\mathrm{PAB})^2$$
$$=16\{(\mathrm{PEF})^2+(\mathrm{PFD})^2+(\mathrm{PDE})^2\}+(\mathrm{ABC})^2$$

(Hind's Trigonometry, 78, page 311).

[6]. Some trigonometrical formulæ may be obtained as follows:—

Since G the centre of gravity of the triangle ABC, Q the centre of its circumscribed circle, and the intersection P of its perpendiculars, lie in one right line, the area GQP vanishes, therefore

$$0=(\mathrm{A}+\mathrm{B}+\mathrm{C})(\sin 2\mathrm{A}.\mathrm{A}+\sin 2\mathrm{B}.\mathrm{B}$$
$$+\sin 2\mathrm{C}.\mathrm{C})(\tan \mathrm{A}.\mathrm{A}+\tan \mathrm{B}.\mathrm{B}+\tan \mathrm{C}.\mathrm{C})$$

$$0=(\sin 2B.AB+\sin 2C.AC+\sin 2A.BA+\sin 2C.BC$$
$$+\sin 2A.CA+\sin 2B.CB)(\tan A.A+\tan B.B+\tan C.C)$$
$$0=\{(\sin 2B-\sin 2C)BC+(\sin 2C-\sin 2A)CA$$
$$+(\sin 2A-\sin 2B)AB\}(\tan A.A+\tan B.B+\tan C.C)$$
$$0=(\sin 2B-\sin 2C)\tan A+(\sin 2C-\sin 2A)\tan B$$
$$+(\sin 2A-\sin 2B)\tan C.$$

[7]. To find the area of the triangle $O_1O_2O_3$.

Since $(-a+b+c)O_1=-aA+bB+cC$, See **43**.

and $(a-b+c)O_2=aA-bB+cC$

and $(a+b-c)O_3=aA+bB-cC$

therefore $8(s-a)(s-b)(s-c)O_1O_2O_3$

$$=(-aA+bB+cC)(aA-bB+cC)(aA+bB-cC)$$
$$=(abAB-acAC+abBA+bcBC$$
$$+caCA-cbCB)(aA+bB-cC)$$
$$=2(bcBC+caCA)(aA+bB-cC)$$
$$=2bcaBCA+2cabCAB=4abcABC,$$

but $(ABC)^2=s(s-a)(s-b)(s-c)$,

hence $\text{area } O_1O_2O_3=\dfrac{abcs}{2\text{ area }ABC}$.

[8]. ABCD is a rectangle, E any point in BC, F any point in CD; show that the rectangle ABCD is equal to twice the triangle AEF, together with the rectangle BE.DF.

Let $AB=a$, $BC=b$, $BE=x$, $FD=y$

then $aF=yC+(a-y)D$

and $bE=xC+(b-x)B$

hence $abAEF=A\{xC+(b-x)B\}\{yC+(a-y)D\}$

$$=x(a-y)ACD+y(b-x)ABC+(b-x)(a-y)ABD$$
$$=x(a-y)\tfrac{1}{2}ab+y(b-x)\tfrac{1}{2}ab+(b-x)(a-y)\tfrac{1}{2}ab$$

$2AEF=x(a-y)+y(b-x)+(b-x)(a-y)=ab-xy$

therefore $ab=2AEF+xy$. Q.E.D.

[9]. If from the ends of AD, one of the oblique sides of a trapezium ABCD, two lines be drawn to the bisection E of the opposite side, the triangle AED thus formed is half the trapezium.

Since $2E = B + C$

therefore

$$2AED = A(B+C)D = (AB+AC)D = ABD + ACD$$
$$= ABD + BCD \text{ (Euclid, book I.)} = ABCD.$$

[10]. The figure EFGH, formed by joining the bisections of the sides AB, BC, CD, DA of a quadrilateral, is a parallelogram whose area is one half the area of the quadrilateral.

Since $2E = A+B, 2F = B+C, 2G = C+D, 2H = D+A$

therefore $2E - 2F + 2G - 2H = 0$, or $E - F + G - H = 0$

hence EFGH is a parallelogram

and
$$\text{area } EFGH = 2EFG = 2\frac{A+B}{2}\,\frac{B+C}{2}\,\frac{C+D}{2}$$
$$= \tfrac{1}{4}(AB+AC+BC)(C+D)$$
$$= \tfrac{1}{4}(ABC+ABD+ACD+BCD)$$
$$= \tfrac{1}{4}(ABC+ACD) + \tfrac{1}{4}(ABD+BCD)$$
$$= \tfrac{1}{2} \text{ area of the quadrilateral.}$$

[11]. The middle points X, Y, Z of the diagonals AC, BD, FG of a complete quadrilateral lie in one right line. See figure of article 79.

$$8 \text{ area } XYZ = (A+C)(B+D)(F+G)$$
$$= (AB+AD+CB+CD)(F+G)$$
$$= ABF + ABG + ADF + ADG + CBF + CBG + CDF + CDG$$
$$= 0 + ABG + ADF + 0 + CBF + 0 + 0 + CDG$$
$$= (ABG - CGD) - (AFD - CBF)$$
$$= ABCD - ABCD = 0.$$

139. *If*

$$(p+q+r)\,P=pA+qB+rC$$
$$(l+m+n)\,Q=lA+mB+nC$$
$$(e+f+g)\,R=eA+fB+gC,$$

then

$$(p+q+r)(l+m+n)(e+f+g)\ \textit{area}\ PQR$$
$$=\begin{vmatrix} p, & q, & r \\ l, & m, & n \\ e, & f, & g \end{vmatrix}\ \textit{area}\ ABC$$
$$=\{p(mg-nf)+q(ne-lg)+r(lf-me)\}\ \textit{area}\ ABC.$$

The solution is similar to those of examples [6] and [7]. If the Article 139 were the first deduction of Art. 138, the examples [1], [3], [4], [6], [7], would be immediately solved by means of Article 139.

CHAPTER VII.

DISTANCE BETWEEN TWO POINTS.

140. *If* $pP=(x+y+z+\text{etc.})P$

$$=xA+yB+zC+uD+\dots$$

and $qQ=(a+b+c+\text{etc.})Q$

$$=aA+bB+cC+dD+\dots$$

then

$$-PQ^2=\left(\frac{x}{p}-\frac{a}{q}\right)\left(\frac{y}{p}-\frac{b}{q}\right)AB^2$$
$$+\left(\frac{x}{p}-\frac{a}{q}\right)\left(\frac{z}{p}-\frac{c}{q}\right)AC^2+\dots$$
$$+\left(\frac{y}{p}-\frac{b}{q}\right)\left(\frac{z}{p}-\frac{c}{q}\right)BC^2+\dots$$
$$+\left(\frac{z}{p}-\frac{c}{q}\right)\left(\frac{u}{p}-\frac{d}{q}\right)CD^2+\dots$$
$$+\text{etc.}$$

S being any point, then, see 123,

pSP is the resultant of xSA, ySB, zSC, etc.

and qQS is the resultant of aAS, bBS, cCS, etc.

therefore SP is the resultant $\frac{x}{p}$SA, $\frac{y}{p}$SB, $\frac{z}{p}$SC, etc.

and QS that of $\frac{a}{q}$AS, $\frac{b}{q}$BS, $\frac{c}{q}$CS

Hence the resultant of SP, QS is the resultant of

$$\left(\frac{x}{p}-\frac{a}{q}\right)SA,\ \left(\frac{y}{p}-\frac{b}{q}\right)SB,\ \left(\frac{z}{p}-\frac{c}{q}\right)SC, \text{ etc.}$$

But the resultant of SP, QS is easily shown by drawing a figure to be PQ in magnitude. Hence, using a well-known theorem in statics:

$$PQ^2=\left(\frac{x}{p}-\frac{a}{q}\right)^2SA^2+\left(\frac{y}{p}-\frac{b}{q}\right)^2SB^2+\left(\frac{z}{p}-\frac{c}{q}\right)^2SC^2+\dots$$
$$+2\left(\frac{x}{p}-\frac{a}{q}\right)\left(\frac{y}{p}-\frac{b}{q}\right)SA.SB\cos ASB$$
$$+2\left(\frac{x}{p}-\frac{a}{q}\right)\left(\frac{z}{p}-\frac{c}{q}\right)SA.SC\cos ASC+\dots$$
$$+2\left(\frac{y}{p}-\frac{b}{q}\right)\left(\frac{z}{p}-\frac{c}{q}\right)SB.SC\cos BSC+\dots$$
$$+\dots$$

but $\quad 2SA.SB\cos ASB=SA^2+SB^2-AB^2$

and $\quad 2SA.SC\cos ASC=SA^2+SC^2-AC^2$

and so on.

Thus

$$PQ^2=\left(\frac{x}{p}-\frac{a}{q}\right)^2SA^2+\left(\frac{y}{b}-\frac{b}{q}\right)^2SB^2+\left(\frac{z}{p}-\frac{c}{q}\right)^2SC^2+\dots$$
$$+\left(\frac{x}{p}-\frac{a}{q}\right)\left(\frac{y}{p}-\frac{b}{q}\right)(SA^2+SB^2-AB^2)$$
$$+\left(\frac{x}{p}-\frac{a}{q}\right)\left(\frac{z}{p}-\frac{c}{q}\right)(SA^2+SC^2-AC^2)+\dots$$
$$+\left(\frac{y}{p}-\frac{b}{q}\right)\left(\frac{z}{p}-\frac{c}{q}\right)(SB^2+SC^2-BC^2)+\dots$$
$$+\dots$$

or $\quad PQ^2=SA^2\left(\frac{x}{p}-\frac{a}{q}\right)\left(\frac{x}{p}-\frac{a}{q}+\frac{y}{p}-\frac{b}{q}+\frac{z}{p}-\frac{c}{q}+\dots\right)$

$$+SB^2\left(\frac{y}{b}-\frac{b}{q}\right)\left(\frac{x}{p}-\frac{a}{q}+\frac{y}{p}-\frac{b}{q}+\frac{z}{p}-\frac{c}{q}+\dots\right).$$
$$+\text{etc.}$$

$$-AB^2\left(\frac{x}{p}-\frac{a}{q}\right)\left(\frac{y}{b}-\frac{b}{q}\right)-AC^2\left(\frac{x}{p}-\frac{a}{q}\right)\left(\frac{z}{p}-\frac{c}{q}\right)-\text{etc.}$$
$$-BC^2\left(\frac{y}{p}-\frac{b}{q}\right)\left(\frac{z}{p}-\frac{c}{q}\right)-\text{etc.}$$
$$-\text{etc.}$$

now $$\frac{x}{p}-\frac{a}{q}+\frac{y}{p}-\frac{b}{q}+\frac{z}{p}-\frac{c}{q}+\text{etc.}$$
$$=\frac{x+y+z+\text{etc.}}{p}-\frac{a+b+c+\text{etc.}}{q}=1-1=0$$

The required result is now obvious.

Examples. [1]. If E, F bisect the sides CA, AB of the triangle ABC, then $EF=\frac{1}{2}BC$.

For $$E=\tfrac{1}{2}A+0B+\tfrac{1}{2}C$$
$$F=\tfrac{1}{2}A+\tfrac{1}{2}B+0C$$
therefore $$-EF^2=(\tfrac{1}{2}-\tfrac{1}{2})(0-\tfrac{1}{2})AB^2+(\tfrac{1}{2}-\tfrac{1}{2})(\tfrac{1}{2}-0)AC^2$$
$$+(0-\tfrac{1}{2})(\tfrac{1}{2}-0)BC^2$$
or $$-EF^2=-\tfrac{1}{4}BC^2 \quad \text{whence, etc.}$$

[2]. To find the length of the bisector AD of the side BC of the triangle ABC.

Since $$A=1.A+0.B+0.C$$
and $$D=0A+\tfrac{1}{2}B+\tfrac{1}{2}C$$
therefore $$-AD^2=(1-0)(0-\tfrac{1}{2})AB^2+(1-0)(0-\tfrac{1}{2})AC^2$$
$$+(0-\tfrac{1}{2})(0-\tfrac{1}{2})BC^2$$
$$AD^2=\tfrac{1}{4}(2AB^2+2AC^2-BC^2)$$

[3]. To find the length of the bisector AD of the interior angle A of the triangle ABC.

Here $$A=1A+0B+0C$$
and $$D=0A+\frac{b}{b+c}B+\frac{c}{b+c}C$$
hence $$-AD^2=(1-0)\left(0-\frac{b}{b+c}\right)AB^2$$
$$+(1-0)\left(0-\frac{c}{b+c}\right)AC^2+\left(0-\frac{b}{b+c}\right)\left(0-\frac{c}{b+c}\right)BC^2$$

$$AD^2=\frac{bc(a+b+c)(b+c-a)}{(b+c)^2}$$

[4]. To find the distance of G, the centre of gravity of ABC, from O the centre of its inscribed circle.

Here $\qquad 3G=A+B+C$

and $\qquad 2sO=aA+bB+cC$

or to avoid fractions, $6sG=2sA+2sB+2sC$

and $\qquad 6sO=3aA+3bB+3cC$

hence $-36s^2GO^2=(2s-3a)(2s-3b)AB^2$

$\qquad +(2s-3a)(2s-3c)AC^2+(2s-3b)(2s-3c)BC^2$

from which we get

$$GO^2=\tfrac{1}{3}(ab+bc+ca)-\frac{1}{9}(a^2+b^2+c^2)-\frac{2abc}{a+b+c}$$

[5]. To find the length of the line joining the middle points E, F of two opposite sides AB, CD of a quadrilateral ABCD.

$$2E=1A+1B+0C+0D$$
$$2F=0A+0B+1C+1D$$

therefore

$$\begin{aligned}-4EF^2=&(1-0)(1-0)AB^2+(1-0)(0-1)AC^2\\&+(1-0)(0-1)AD^2+(1-0)(0-1)BC^2\\&+(1-0)(0-1)BD^2+(0-1)(0-1)CD^2\end{aligned}$$

or $\quad +4EF^2=-AB^2+AC^2+AD^2+BC^2+BD^2-CD^2$

[6]. In any quadrilateral, the sum of the squares of the diagonals AC, BD, together with four times the square of the line joining their middle points E, F, is equal to the sum of the squares of all the sides.

$$2E=1A+0B+1C+0D$$
$$2F=0A+1B+0C+1D$$

therefore

$$\begin{aligned}-4EF^2 &= (1-0)(0-1)AB^2+(1-0)(1-0)AC^2\\ &\quad +(1-0)(0-1)AD^2+(0-1)(1-0)BC^2\\ &\quad +(0-1)(0-1)BD^2+(1-0)(0-1)CD^2\end{aligned}$$

$$4EF^2=AB^2-AC^2+AD^2+BC^2-BD^2+CD^2$$

whence, etc.

[7]. G being the middle point of the line joining the middle points of the diagonals of the quadrilateral ABCD, to find GA.

$$4G=1A+1B+1C+1D$$
$$A=1A+0B+0C+0D$$

or to avoid fractions,

$$4G=1A+1B+1C+1D$$
$$4A=4A+0B+0C+0D$$

therefore

$$\begin{aligned}-16AG^2 &= (1-4)(1-0)AB^2+(1-4)(1-0)AC^2\\ &\quad +(1-4)(1-0)AD^2+(1-0)(1-0)BC^2\\ &\quad +(1-0)(1-0)BD^2+(1-0)(1-0)CD^2\end{aligned}$$

$$16AG^2=3(AB^2+AC^2+AD^2)-(BC^2+BD^2+CD^2)$$

CHAPTER VIII.

MULTIPLICATION BY P^2-Q^2, BY P^2, BY A CIRCLE, BY CIRCLE — CIRCLE, BY P^2— CIRCLE.—SQUARING.— DISTANCE BETWEEN TWO POINTS.

MULTIPLICATION BY P^2-Q^2.

141. *If* $(a+b+c+\ldots)G=aA+bB+cC+\ldots$ (1)
and P, Q *be any two points, then*

$$(a+b+c+\text{etc.})G(P^2-Q^2) = (aA+bB+cC+\ldots)(P^2-Q^2)$$

where $aA(P^2-Q^2)=aAP^2-aAQ^2$, AP^2 *meaning the square on* AP.

From (1) we derive as a true equation of forces:
$(a+b+c+\text{etc.})PG=aPA+bPB+cPC+\text{etc.}$ See 123.

Resolve along PQ, then

$$(a+b+c+\text{etc.})PG\cos GPQ = aPA\cos APQ+bPB\cos BPQ+\text{etc.}$$

multiply by 2PQ, and remark that

$$2PG.PQ\cos GPQ=GP^2+PQ^2-GQ^2$$

therefore $(a+b+c+\text{etc.})(GP^2+PQ^2-GQ^2)$

$$=a(AP^2+PQ^2-AQ^2)+b(BP^2+PQ^2-BQ^2)+\text{etc.}$$

whence, after easy reductions, we obtain the required result.

142. *Corollary.* Since from

$$(a+b+c+\text{etc.})G=aA+bB+cC+\text{etc.} \qquad (1)$$

a true result is obtained by multiplying both sides by P^2-Q^2; a true result will obviously also be obtained by multiplying both sides of equation (1) by

$$pP^2+qQ^2+rR^2+\ldots$$

provided $p+q+r+\text{etc.}=0$.

Examples. [1]. If a line AB be divided in C, and P be any point, then

$$BC.AP^2+CA.BP^2=BC.AC^2+AC.BC^2+AB.CP^2$$

(Townsend, page 93.)

For $\qquad AB.C=AC.B+BC.A$

multiply by P^2-C^2

then $AB.CP^2=AC.BP^2+BC.AP^2-AC.BC^2-BC.AC^2$

[2]. If O be the mean point of the n points A, B, C, D, etc., in one right line, and OS be perpendicular on AB, and such that

$$n.OS^2=AO^2+BO^2+CO^2+DO^2+\text{etc.}$$

then P being any point in AB or AB produced,

$$nSP^2=SA^2+SB^2+SC^2+SD^2+\text{etc.}$$

Since $\qquad nO=A+B+C+\ldots$

multiply by P^2-O^2

therefore $\qquad nPO^2=PA^2+PB^2+PC^2+PD^2+\ldots$

$$-(AO^2+BO^2+CO^2\ldots)$$

$$=PA^2+PB^2+PC^2+\text{etc.}-nOS^2$$

therefore

$$PA^2+PB^2+PC^2+\text{etc.}=n(PO^2+OS^2)=nPS^2$$

[3]. In any triangle the squares of the two sides AB, AC are together double of the two squares of half the base BC, and of the straight line joining its bisection D with the vertex.

Since $\qquad 2D=B+C$

multiply by $A^2—D^2$

therefore $2AD^2=AB^2+AC^2—BD^2—CD^2$

but $CD=BD$

hence $2AD^2+2BD^2=AB^2+AC^2$.

[4]. G being the centre of gravity of the triangle ABC, $3(GA^2+GB^2—2GC^2)=2c^2—a^2—b^2$

For $3G=A+B+C$

multiply by $A^2+B^2—2C^2$ See 142.

then $3(GA^2+GB^2—2GC^2)$

$=0+AB^2+AC^2+AB^2+0+BC^2—2AC^2—2BC^2—0$

$=2AB^2—AC^2—BC^2$.

[5]. If with any point R as centre, and a radius q equal to that of the circumscribed circle of the triangle ABC, a circle be described and tangents AA′, BB′, CC′, drawn to it, then

$$AA'^2+BB'^2+CC'^2=3(GR^2—GQ^2)$$

For $3G=A+B+C$

Multiply by $R^2—Q^2$, where Q is the centre of the circumscribed circle;

therefore

$3(GR^2—GQ^2)=AR^2+BR^2+CR^2—q^2—q^2—q^2$

(since $AQ=q=BQ=CQ$)

$=(AR^2—q^2)+(BR^2—q^2)+(CR^2—q^2)$

$=AA'^2+BB'^2+CC'^2$

[6]. In the triangle ABC, to prove

$$(a^2—b^2)\sin 2C=c^2(\sin 2B—\sin 2A)$$

Since

(...) $Q=\sin 2A\,.\,A+\sin 2B\,.\,B+\sin 2C\,.\,C$ See 31.

$\times A^2—B^2$

therefore, remarking that $QA=QB$

$0=\sin 2B.AB^2+\sin 2C.AC^2—\sin 2A\,.\,AB^2—\sin 2C.BC^2$

[7]. To prove

$$0=\left(\frac{1}{b}-\frac{1}{c}\right)OA^2+\left(\frac{1}{c}-\frac{1}{a}\right)OB^2+\left(\frac{1}{a}-\frac{1}{b}\right)OC^2$$

where O is the centre of the circle inscribed in ABC.

Here $(a+b+c)\ O=aA+bB+cC$

$$\times\left(\frac{1}{b}-\frac{1}{c}\right)A^2+\left(\frac{1}{c}-\frac{1}{a}\right)B^2+\left(\frac{1}{a}-\frac{1}{b}\right)C^2. \qquad \text{See 142.}$$

hence

$$(a+b+c)\left\{\left(\frac{1}{b}-\frac{1}{c}\right)OA^2+\left(\frac{1}{c}-\frac{1}{a}\right)OB^2+\left(\frac{1}{a}-\frac{1}{b}\right)OC^2\right\}$$

$$=\left(\frac{1}{b}-\frac{1}{c}\right)(bc^2+cb^2)+\left(\frac{1}{c}-\frac{1}{a}\right)(ac^2+ca^2)$$

$$+\left(\frac{1}{a}-\frac{1}{b}\right)(ab^2+ba^2)=0$$

[8]. In a parallelogram, ABCD, the squares of the diagonals are equal to the sum of the squares of the sides.

Here $\quad 0=A-B+C-D$

$\times A^2-B^2+C^2-D^2. \qquad$ See 142.

therefore

$$0=-2AB^2+2AC^2-2AD^2-2BC^2+2BD^2-2CD^2$$

hence $AC^2+BD^2=AB^2+BC^2+CD^2+DA^2$.

[9]. The squares of the diagonals of a trapezium ABCD, are together equal to the squares of its two oblique sides, with twice the rectangle contained by its parallel sides AB, CD.

Since CD, AB are parallel, $CD\ (A-B)=AB\ (D-C)$

or $0=CD\,.\,A-CD.B+AB\,.\,C-AB\,.\,D$

$\times CD\,.\,A^2-CD\,.\,B^2+AB\,.\,C^2-AB\,.\,D^2. \qquad$ See 142.

Therefore

$$\begin{aligned}0=&-CD^2.\,AB^2.\ +AB\,.\,CD\,.\,CA^2-AB\,.\,CD\,.\,AD^2\\&-CD^2.\,AB^2-CD\,.\,AB\,.\,BC^2+CD\,.\,AB\,.\,BD^2\\&+CD\,.\,AB\,.\,AC^2-CD\,.\,AB\,.\,BC^2-AB^2.\,CD^2\\&-AB\,.\,CD\,.\,AD^2+CD\,.\,AB\,.\,BD^2-AB^2.\,CD^2\end{aligned}$$

Divide by 2AB.CD,

then $0=-2AB\,.\,CD+CA^2-AD^2-BC^2+BD^2$.

Squaring.

143. From the last two examples, it is obvious that *When* $0=aA+bB+cC+$etc., *then*

$$0=(aA+bB+cC+\text{etc.})(aA^2+bB^2+cC^2+\text{etc.})$$

For shortness sake, I shall denote by $(aA+bB+cC+\text{etc.})^2$ *the product* $(aA+bB+cC+\text{etc.})(aA^2+bB^2+cC^2+\text{etc.})$

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[10]. If from the right angle C of a right angled triangle, lines be drawn to the opposite angles of any rectangle AEDB on the hypothenuse AB, the difference of the squares on these lines is equal to the difference of the squares on the two sides of the triangle.

AEDB being a rectangle, and therefore a parallelogram, we have

$$A+D=B+E$$
$$\times\, O^2-C^2$$

Where O is the intersection of the diagonals of the rectangle,

then $AO^2+DO^2-AC^2-DC^2=BO^2+EO^2-BC^2-EC^2$

but AEDB being a rectangle $OA=OB=OD=OE$

hence $CD^2-CE^2=CB^2-CA^2$.

[11]. A, B, C, D, etc., are fixed points in one plane; to find the locus of a point P which moves so that $aPA^2+bPB^2+cPC^2+$etc.$=$constant; a, b, c, etc., being constants.

Let G be the fixed point

$$(a+b+c+\text{etc.})\ G=aA+bB+cC+\text{etc.}$$
$$\times\, P^2-G^2$$

therefore $(a+b+c+\text{etc.})\ GP^2$

$$=(aAP^2+bBP^2+\text{etc.})-(aAG^2+bBG^2+\text{etc.})$$

Hence GP is constant, and G being a fixed point, the locus of P is a circle centre G.

[12]. A point O is taken within the equilateral triangle ABC, such that area OBC : OCA : OAB= $p:q:r$, to prove

$$(q-r)OA^2+(r-p)OB^2+(p-q)OC^2=0$$

Here $(p+q+r)O=pA+qB+rC$ See 29.

$\times(q-r)A^2+(r-p)B^2+(p-q)C^2$ See 142.

then $(p+q+r)\{(q-r)OA^2+(r-p)OB^2+(p-q)OC^2\}$

$$=(q-r)(qAB^2+rAC^2)+(r-p)(pAB^2+rBC^2)$$
$$+(p-q)(pAC^2+qBC^2)=0$$

since here AB=AC=BC.

---

144. *If* $0=aA+bB+cC+\text{etc.}$ (1)

then

$$(aA+bB+cC+\text{etc.})A^2=(aA+bB+cC+\text{etc.})B^2=\text{etc.}$$
$$=(aA+bB+cC+\text{etc.})P^2$$

where P *is any point.*

For equation (1) gives $-aA=bB+cC+\text{etc.}$

$\times P^2-A^2$

therefore

$$-aAP^2=bBP^2+cCP^2+\text{etc.}-(bB+cC+\text{etc.})A^2$$

or $(aA+bB+cC+\text{etc.})A^2=(aA+bB+cC+\text{etc.})P^2$.

Examples. [1]. If D bisects the base BC of the triangle ABC, $2AD^2+2BD^2=AB^2+AC^2$.

For $0=2D-B-C$

therefore $(2D-B-C)A^2=(2D-B-C)D^2$

$2DA^2-AB^2-AC^2=-BD^2-CD^2=-2BD^2$, hence, etc.

[2]. G being the CG of the triangle ABC, and P any point, $AP^2+BP^2+CP^2-3GP^2=\text{constant}$.

For $0=A+B+C-3G$
therefore $(A+B+C-3G)P^2=(A+B+C-3G)G^2$
whence
$AP^2+BP^2+CP^2-3GP^2=AG^2+BG^2+CG^2=\text{constant}.$

[3]. The sum of the squares of the diagonals of a parallelogram ABCD is equal to the sum of the squares of its sides.

For $0=A-B+C-D$
therefore $(A-B+C-D)A^2=(A-B+C-D)B^2$
hence $-AB^2+CA^2-AD^2=AB^2+BC^2-BD^2$
therefore $AC^2+BD^2=AB^2+BC^2+CD^2+DA^2$
since $CD=AB$.

MULTIPLICATION BY P^2.

145. *If* $(a+b+c+\text{etc.})G=aA+bB+cC+\text{etc.}$ (1) *and* P *is any point, then*

$$(a+b+c+\text{etc.})^2GP^2$$
$$=(a+b+c+\text{etc.})(aAP^2+bBP^2+cCP^2+\text{etc.})$$
$$-\tfrac{1}{2}(aA+bB+cC+\text{etc.})^2$$ See 143.

From (1) results the following true equation of forces,
$(a+b+c+\text{etc.})PG=aPA+bPB+cPC+\text{etc.}$ See 123.
equating values of the square of the resultant of the forces aPA along PA, bPB along PB, etc., therefore

$$(a+b+c+\text{etc.})^2PG^2=a^2PA^2+b^2PB^2+c^2PC^2+\text{etc.}$$
$$+2abPA.PB\cos APB+2acPA.PC\cos APC+\text{etc.}$$
$$+2bcPB.PC\cos BPC+\text{etc.}$$
$$+\text{etc.}$$
$$=a^2PA^2+b^2PB^2+c^2PC^2+\text{etc.}$$
$$+ab(PA^2+PB^2-AB^2)+ac(PA^2+PC^2-AC^2)+\text{etc.}$$
$$+bc(PB^2+PC^2-BC^2)$$
$$+\text{etc.}$$

or $$(a+b+c+\text{etc.})^2PG^2$$
$$=(a+b+c+\text{etc.})(aPA^2+bPB^2+cPC^2+\text{etc.})$$
$$-(abAB^2+acAC^2+\text{etc.}+bcBC^2+\text{etc.})$$

I shall call this equation the result of the multiplication of $(a+b+c+\text{etc.})G=aA+bB+cC+\text{etc.}$ *by* P^2.

Examples. [1]. If CD be drawn from the vertex C to any point D in the base of the triangle ABC, then

$$AC^2.BD+BC^2.AD=CD^2.AB+AB.AD.BD$$

We have $$AB.D=AD.B+BD.A$$
$$\times C^2$$

then $AB^2.CD^2=AB(AD.BC^2+BD.AC^2)-AD.BD.AB^2$ dividing by AB and transposing we obtain the required result.

[2]. To prove

$$3(GA^2+GB^2+GC^2)=BC^2+CA^2+AB^2$$

in the triangle ABC.

Since $$3G=A+B+C$$
$$\times G^2$$

therefore $0=3(GA^2+GB^2+GC^2)-(AB^2+AC^2+BC^2)$

[3]. To prove $GA^2=\frac{1}{9}(2b^2+2c^2-a^2)$ in the triangle ABC.

Since $$3G=A+B+C$$
$$\times A^2$$

hence $9GA^2=3(AB^2+AC^2)-(AB^2+AC^2+BC^2)$
$$9GA^2=2AB^2+2AC^2-BC^2$$

[4]. To prove $GQ^2=q^2-\frac{1}{9}(a^2+b^2+c^2)$ where G is the CG of ABC ; Q, q the centre and radius of its circumscribed circle.

Since $$3G=A+B+C$$
$$\times Q^2$$

hence $9GQ^2=3(q^2+q^2+q^2)-(AB^2+AC^2+BC^2)$

[5]. Given $(x+y+z)P=xA+yB+zC$ (1) to find PQ^2; and determine the equation uniting x, y, z when P lies on the circumscribed circle.

(i). Multiply (1) by Q^2, therefore

$$(x+y+z)^2PQ^2=(x+y+z)(xq^2+yq^2+zq^2) \\ -(yzBC^2+zxCA^2+xyAB^2)$$

or $$PQ^2=q^2-\frac{yza^2+zxb^2+xyc^2}{(x+y+z)^2}$$

(ii). When P lies on the circumscribed circle $PQ=q$, and the required condition between x, y, z is

$$0=yza^2+zxb^2+xyc^2 \quad \text{or} \quad 0=\frac{a^2}{x}+\frac{b^2}{y}+\frac{c^2}{z}$$

(iii). The geometrical meaning of this equation is (when AP crosses BC and not BC produced, that is, when x is negative),

$$0=-\frac{a^2}{\text{area PBC}}+\frac{b^2}{\text{PCA}}+\frac{c^2}{\text{PAB}}$$

Let PD, PE, PF be perpendiculars on BC, CA, AB, and the last equation becomes

$$\frac{BC}{PD}=\frac{CA}{PE}+\frac{AB}{PF} \quad \text{or} \quad \frac{\sin A}{PD}=\frac{\sin B}{PE}+\frac{\sin C}{PF}$$

whence $\frac{1}{2}PE.PF \sin A=\frac{1}{2}PD.PE \sin C+\frac{1}{2}PD.PF \sin B$ or area PEF=area PED+area PDF; whence D, E, F lie in one right line. Derivable from Art. 133.

Or, the feet of the perpendiculars from any point of the circumscribed circle upon the sides of the triangle ABC lie in one right line.

[6]. An expression for the radius q of the circumscribed circle.

$$(\ldots\ldots)Q=\sin 2A.A+\sin 2B.B+\sin 2C.C$$
$$\times Q^2$$

$$0=(\sin 2A+\sin 2B+\sin 2C)(\sin 2A.q^2+\sin 2B.q^2 +\sin 2C.q^2)$$
$$-(\sin 2B \sin 2C.a^2+\sin 2C \sin 2A.b^2 +\sin 2A \sin 2B.c^2)$$

or $$(\sin 2A+\sin 2B+\sin 2C)q=(\sin 2B \sin 2Ca^2 +\sin 2C \sin 2Ab^2+\sin 2A \sin 2Bc^2).$$

[7]. AD is the bisector of the angle BAC of the triangle ABC; to prove

$$AB.AC=AD^2+BD.DC$$

Here $$(b+c)D=bB+cC$$ See 19.

$$\times A^2$$

hence $$(b+c)^2AD^2=(b+c)(bc^2+cb^2)-bca^2$$

or $$AD^2=bc-\frac{bca^2}{(b+c)^2}=bc-\frac{ba}{b+c}\cdot\frac{ca}{b+c}$$
$$=AC.AB-CD.BD$$

[8]. To prove $aAO^2+bBO^2+cCO^2=abc$ in the triangle ABC.

Since $$(a+b+c)O=aA+bB+cC$$ See 37.

$$\times O^2$$

then $$0=(a+b+c)(aAO^2+bBO^2+cCO^2) -(bca^2+cab^2+abc^2)$$

dividing by $a+b+c$ the required result is obtained.

[9]. To find OA^2.

Since $$(a+b+c)O=aA+bB+cC$$

multiplying by A^2, therefore

$$(a+b+c)^2OA^2=(a+b+c)(bc^2+cb^2)-(abc^2+acb^2+bca^2)$$

hence $$(a+b+c)OA^2=bc(b+c-a)$$

[10]. If the centre O of the inscribed circle of a triangle ABC be fixed, and α, β, γ represent the distances of its angles from any other fixed point P, then will $a\alpha^2+b\beta^2+c\gamma^2$ be invariable for all situations of the triangle, of which the sides are given.

Here $\quad (a+b+c)O=aA+bB+cC$
$$\times P^2$$
therefore $(a+b+c)^2OP^2=(a+b+c)(a\alpha^2+b\beta^2+c\gamma^2)$
$$-(abc^2+acb^2+bca^2)$$
or $\quad (a+b+c)OP^2=a\alpha^2+b\beta^2+c\gamma^2-abc$

therefore $a\alpha^2+b\beta^2+c\gamma=abc+(a+b+c)OP^2=\text{constant}.$

[11]. To prove $QO^2=q^2-2qr$ where q, r are the radii of the circumscribed and inscribed circles.
$$(a+b+c)O=aA+bB+cC$$
$$\times Q^2$$
therefore $(a+b+c)^2OQ^2=(a+b+c)(aq^2+bq^2+cq^2)$
$$-(abc^2+acb^2+bca^2)$$
or $\quad (a+b+c)OQ^2=(a+b+c)q^2-abc$
$$OQ^2=q^2-\frac{abc}{a+b+c}=q^2-2\frac{abc}{4\text{ area ABC}}\ \frac{2\text{ area ABC}}{a+b+c}$$
hence $\quad OQ^2=q^2-2qr.$

[12]. Similarly $O_1Q^2=q^2+2qr_1$, where r_1 is the radius of the escribed circle touching BC.

[13]. To prove $OO_1^2+OO_2^2+OO_3^2=16q^2-8qr$, where q, r are the radii of the circumscribed and inscribed circles.

Since $\quad 4Q=O+O_1+O_2+O_3$ See 54.
$$\times O^2-Q^2 \qquad \text{See 141.}$$
hence
$$4QO^2=OO_1^2+OO_2^2+OO_3^2-QO^2-QO_1^2-QO_2^2-QO_3^2$$
then $\quad OO_1^2+OO_2^2+OO_3^2=5(q^2-2qr)+(q^2+2qr_1)$
$$+(q^2+2qr_2)+(q^2+2qr_3)$$
$$=8q^2+2q(r_1+r_2+r_3-r)-8qr$$
$$=8q^2+2q.4q-8qr=16q^2-8qr$$

[14]. To find the sum of the squares on the sides of the triangle $O_1O_2O_3$.

Since $\quad 4Q=O+O_1+O_2+O_3$ See 54.

therefore $\quad 4Q-O=O_1+O_2+O_3=3M \quad$ suppose
$$\times Q^2$$
$$9MQ^2=3(O_1Q^2+O_2Q^2+O_3Q^2)-\text{required sum}$$
$$=3(-OQ^2)+4OQ^2$$
required sum
$$=3(q^2+2qr_1+q^2+2qr_2+q^2+2qr_3)-(q^2-2qr)$$
$$=8q^2+6q(r_1+r_2+r_3-r)+8qr$$
$$=8q^2+6q.4q+8qr$$
$$=32q^2+8qr.$$

[15]. The sum of the squares of all the lines uniting the centres of the inscribed and escribed circles is $48q^2$.

Since $\quad 4Q=O+O_1+O_2+O_3 \quad$ See 54.
$$\times Q^2$$
therefore
$$0=4(OQ^2+O_1Q^2+O_2Q^2+O_3Q^2)-\text{required sum}$$
required sum
$$=4(q^2-2qr+q^2+2qr_1+q^2+2qr_2+q^2+2qr_3)$$
$$=16q^2+8q(r_1+r_2+r_3-r)$$
$$=16q^2+8q.4q=48q^2.$$

[16]. Prove that the square of any straight line AD, drawn from the vertex A of an isosceles triangle ABC to the base BC, is less than the square of a side of the triangle by the rectangle contained by the segments of the base.

Since $\quad BC.D=BD.C+CD.B$
$$\times A^2$$
$$BC^2.DA^2=BC(BD.CA^2+CD.BA^2)-BD.CD.BC^2$$
hence $BC.DA^2=BD.CA^2+CD.BA^2-BD.CD.BC$

but $AB=AC$, therefore
$$BC.DA^2=AB^2(BD+CD)-BD.CD.BC$$
or $\quad DA^2=AB^2-BD.CD$

[17]. If from any point P, within a rectangle ABCD, lines be drawn to the angular points, the sums of the squares upon those drawn to the opposite angles will be equal.

Since a rectangle is a parallelogram, therefore

$$D=A-B+C \qquad \text{See 90.}$$

$$\times P^2$$

$$DP^2=AP^2-BP^2+CP^2+AB^2-AC^2+BC^2$$

but $AC^2=AB^2+BC^2$ since ABC is a right angle.

hence $\qquad DP^2+BP^2=PA^2+PC^2.$

[18]. G being the mean point of a quadrilateral ABCD, it is required to find GA^2.

$$4G=A+B+C+D$$

$$\times A^2$$

$$16GA^2=4(BA^2+CA^2+DA^2)$$
$$-(AB^2+AC^2+AD^2+BC^2+BD^2+CD^2)$$

hence

$$16GA^2=3(BA^2+CA^2+DA^2)-(BC^2+BD^2+CD^2).$$

[19]. ABCD etc., is a regular polygon of n sides, whose centre is O, and r is the radius of its circumscribed circle. To prove that the sum of the squares of all the different lines obtained by joining the angular points of the polygon is n^2r^2.

O being the mean point of the polygon,

$$nO=A+B+C+\text{etc.}$$

$$\times O^2$$

$$0=n(AO^2+BO^2+CO^2+\text{etc.})-\text{the required sum.}$$

but $\qquad AO=BO=CO=\text{etc.}=r$

hence $\qquad 0=n^2r^2-$the required sum.

[20]. If two points A, B be taken in the diameter of a circle, equally distant from the centre O, the sum of the squares of the two lines drawn from those points

to any point P in the circumference will be constant.

Here $2O=A+B$

$\times P^2$

$4OP^2=2(PA^2+PB^2)-AB^2$, therefore

$PA^2+PB^2=\frac{1}{2}AB^2+2OP^2=\frac{1}{2}AB^2+2$ square on radius

$=$constant.

[21]. If from the centre C of a circle, a line be drawn to any point D in the chord AB of an arc, the square of that line, together with the rectangle of the segments of the chord, will be equal to the square of the radius *r*.

Here $AB.D=BD.A+AD.B$

$\times C^2$

therefore $AB^2.CD^2=AB.BD.AC^2+AB.AD.BC^2$

$-BD.AD.AB^2$

$$AB.CD^2=BD.r^2+AD.r^2-BD.AD.AB$$
$$=r^2.AB-BD.AD.AB$$
$$CD^2=r^2-BD.AD.$$

[22]. A point A is taken in the radius of a circle, and another point B in the same radius produced, so that the radius *r* is a mean proportional between their distances from the centre O; prove that the lines drawn from A, B to any the same point P in the circumference are always in the same ratio.

Since $OB.A=OA.B+(OB-OA)O$

multiplying by OA, since $OA.OB=r^2$

$$r^2.A=OA^2.B+(r^2-OA^2)O$$

multiply by P^2

$r^4AP^2=r^2OA^2BP^2+r^2(r^2-OA^2)OP^2-OA^2(r^2-OA^2)OB^2$

now $OP=r$, and $OA.OB=r^2$, dividing by r^2 we get

$r^2.AP^2=OA^2.BP^2+r^4-r^2.OA^2-r^2(r^2-OA^2)=OA^2.BP^2$

hence $AP : BP=OA : r=$constant ratio.

[23]. Two circles, centres A, B, radii a, b, touch externally in C, and are touched by DE in the points D, E; show that $ED^2 : CD^2 : CE^2 = a+b : a : b$

$$(a+b)C = aB + bA$$
$$\times D^2$$
$$(a+b)^2CD^2 = (a+b)(aBD^2 + bAD^2) - abAB^2$$
$$= (a+b)\{a(BE^2+DE^2) + ba^2\} - ab(a+b)^2$$
$$(a+b)CD^2 = ab^2 + aDE^2 + ba^2 - ab(a+b) = aDE^2$$

similarly the multiplication by E^2 gives $(a+b)CE^2 = bDE^2$ whence the required proportion. Also $ED^2 = CE^2 + CD^2$, therefore the angle DCE is a right angle.

Multiplication by P^2+Q^2+ etc.

146. When $(a+b+c+\text{etc.})G = aA + bB + cC + \text{etc.}$
$$\times P^2$$
we get $(a+b+c+\text{etc.})^2GP^2$
$$= (a+b+c+\text{etc.})(aAP^2 + bBP^2 + \text{etc.})$$
$$-(ab.AB^2 + acAC^2 + \text{etc.} + bcBC^2 + \text{etc.})$$
Similarly $(a+b+c+\text{etc.})^2GQ^2$
$$= (a+b+c+\text{etc.})(aAQ^2 + bBQ^2 + \text{etc.})$$
$$-(ab.AB^2 + acAC^2 + \text{etc.})$$ and so on.

Hence, n being the number of points P, Q, R,...
$$(a+b+c+\text{etc.})^2G(P^2+Q^2+R^2+\text{etc.}) =$$
$$(a+b+c+\text{etc.})(aA+bB+cC+\text{etc.})(P^2+Q^2+R^2+\text{etc.})$$
$$-n(abAB^2 + acAC^2 + \text{etc.} + bcBC^2 + \text{etc.})$$

Examples. [1]. G being the CG of the triangle ABC,
$$GA^2+GB^2+GC^2 = \tfrac{1}{3}(BC^2+CA^2+AB^2)$$
For $3G = A+B+C$
$$\times A^2+B^2+C^2$$

then $9(GA^2+GB^2+GC^2)$

$$=3(AB^2+AC^2+AB^2+BC^2+AC^2+BC^2)$$
$$-3(BC^2+CA^2+AB^2)$$
$$3(GA^2+GB^2+GC^2)=BC^2+CA^2+AB^2.$$

[2]. To find the sum of the squares of the distances of the centre O of the circle inscribed in the triangle ABC from the vertices A, B, C.

Since $\quad (a+b+c)O=aA+bB+cC$

$$\times A^2+B^2+C^2$$
$$(a+b+c)^2(OA^2+OB^2+OC^2)$$
$$=(a+b+c)(ac^2+ab^2+bc^2+ba^2+cb^2+ca^2)$$
$$-3(bca^2+cab^2+abc^2)$$

or $\quad (a+b+c)(OA^2+OB^2+OC^2)$

$$=a(b^2+c^2)+b(c^2+a^2)+c(a^2+b^2)-3abc$$
$$=(ab+bc+ca)(a+b+c)-6abc.$$

[3]. $(-a+b+c)(O_1A^2+O_1B^2+O_1C^2)$

$$=6abc+(bc-ca-ab)(-a+b+c).$$

This result differs from that in Example [2] by the sign of a.

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147. Let $(a+b+c+\text{etc.})G=aA+bB+cC+\text{etc.})$ (1) the number of points in the second member being n, then multiplying by $A^2+B^2+C^2+$ etc., therefore

$$(a+b+c+\text{etc.})^2(GA^2+GB^2+GC^2+\text{etc.})$$
$$=(a+b+c+\text{etc.})(aA+bB+\text{etc.})(A^2+B^2+\text{etc.})$$
$$-n(abAB^2+acAC^2+\text{etc.}+bcBC^2+\text{etc}) \qquad (2)$$

but (1) multiplied by G^2 gives

$$0=(a+b+c+\text{etc.})(aA+bB+cC+\text{etc.})G^2$$
$$-(abAB^2+acAC^2+\text{etc.}+bcBC^2+\text{etc.}) \qquad (3)$$

from (2) and (3)

$$(a+b+c+\text{etc.})G(A^2+B^2+C^2+\text{etc.})$$
$$=(aA+bB+cC+\text{etc.})(A^2+B^2+C^2+\text{etc.})$$
$$-n(aA+bB+cC+\text{etc.})G^2$$

or $(aA+bB+cC+\text{etc.})(A^2+B^2+C^2+\text{etc.})$
$$=(a+b+c+\text{etc.}+na)GA^2$$
$$+(a+b+c+\text{etc.}+nb)GB^2+\text{etc.}$$

Example. $(a+b+c)O=aA+bB+cC$ therefore
$$(aA+bB+cC)(A^2+B^2+C^2)$$
$$=(a+b+c+3a)OA^2+(a+b+c+3b)OB^2$$
$$+(a+b+c+3c)A^2$$
or $a(b^2+c^2)+b(c^2+a^2)+c(a^2+b^2)$
$$=(4a+b+c)OA^2+(a+4b+c)OB^2+(a+b+4c)OC^2.$$

MULTIPLICATION BY $pP^2+qQ^2+rR^2+$ ETC.

148. *If* $(a+b+c+\text{etc.})G=aA+bB+cC+\ldots$ *multiplying by* $pP^2+qQ^2+rR^2+\text{etc.}$, *in which the number of points really is* $p+q+r+\text{etc.}$; *then*
$$(a+b+c+\text{etc.})^2G(pP^2+qQ^2+rR^2+\text{etc.})$$
$$=(a+b+c+\text{etc.})(aA+bB+cC+\text{etc.})(pP^2+qQ^2+\text{etc.})$$
$$-(p+q+r+\text{etc.})(abAB^2+acAC^2+\text{etc.}+bcBC^2+\text{etc.})$$

Examples. [1]. G being the CG of a triangle, to find the value of $2GA^2+2GB^2-GC^2$.

Since $3G=A+B+C$
$$\times\, 2A^2+2B^2-C^2$$
hence $9(2GA^2+2GB^2-GC^2)$
$$=3(A+B+C)(2A^2+2B^2-C^2)-3(BC^2+CA^2+AB^2)$$
$$=3(2c^2-b^2+2c^2-a^2+2b^2+2a^2)-3(a^2+b^2+c^2)$$
or $2GA^2+2GB^2-GC^2=AB^2$.

[2]. O being the centre of the inscribed circle of the triangle ABC, to prove $aOA^2+bOB^2+cOC^2=abc$.

Since $(a+b+c)O=aA+bB+cC$
$\times\, aA^2+bB^2+cC^2$

then $(a+b+c)^2(aOA^2+bOB^2+cOC^2)$
$=(a+b+c)(aA+bB+cC)(aA^2+bB^2+cC^2)$
$-(a+b+c)(bca^2+cab^2+abc^2)$

$(a+b+c)(aAO^2+bBO^2+cCO^2)$
$=abc^2+acb^2+abc^2+bca^2+acb^2+bca^2$
$-bca^2-cab^2-abc^2$
$=abc(a+b+c)$

therefore, etc.

[3]. O being the centre of the inscribed circle of the triangle ABC, to prove

$$(b+c)OA^2+(c+a)OB^2+(a+b)OC^2$$
$$=a^2(b+c)+b^2(c+a)+c^2(a+b)-4abc.$$

Distance between Two Points.

149. *If* $P=aA+bB+cC+\text{etc.}$ (1)
where of course $1=a+b+c+\text{etc.}$
and $Q=xA+yB+zC+\text{etc.}$ (2)
where $1=x+y+z+\text{etc.}$

then $PQ^2=(aA+bB+cC+\text{etc.})(xA^2+yB^2+zC^2+\text{etc.})$
$-\frac{1}{2}(aA+bB+cC+\text{etc.})^2$
$-\frac{1}{2}(xA+yB+zC+\text{etc.})^2.$ See 143.

For multiplying (1) by Q^2, we get See 145.

$$PQ^2=(aA+bB+cC+\text{etc.})Q^2$$
$$-\tfrac{1}{2}(aA+bB+cC+\text{etc.})^2 \quad (3)$$

again, multiplying (2) by $aA^2+bB^2+cC^2+\text{etc.}$ See 148.

$(aA+bB+cC+\text{etc.})Q^2$
$=(aA+bB+cC+\text{etc.})(xA^2+yB^2+zC^2+\text{etc.})$
$-(a+b+c+\text{etc.})\{\frac{1}{2}(xA+yB+zC+\text{etc.})^2\}.$ (4)

but $a+b+c+\text{etc.}=1$
therefore by adding (3) and (4) the required result is obtained.

Examples. [1]. G being the centre of gravity of the triangle ABC, to find GA.

Since $\quad G=\frac{1}{3}A+\frac{1}{3}B+\frac{1}{3}C$

and $\quad A=A$

hence

$$GA^2=(\tfrac{1}{3}A+\tfrac{1}{3}B+\tfrac{1}{3}C)A^2-\tfrac{1}{2}(\tfrac{1}{3}A+\tfrac{1}{3}B+\tfrac{1}{3}C)^2-\tfrac{1}{2}(A)^2$$

$$=\tfrac{1}{3}(c^2+b^2)-\tfrac{1}{9}(a^2+b^2+c^2)-0$$

or $\quad GA^2=\frac{1}{9}(2b^2+2c^2-a^2)$

[2]. To find the distance of O, the centre of the inscribed circle, from the middle point D of BC.

$$O=\frac{a}{2s}A+\frac{b}{2s}B+\frac{c}{2s}C$$

$$D=\tfrac{1}{2}B+\tfrac{1}{2}C, \text{ hence}$$

$$OD^2=\left(\frac{a}{2s}A+\frac{b}{2s}B+\frac{c}{2s}C\right)(\tfrac{1}{2}B^2+\tfrac{1}{2}C^2)$$

$$-\tfrac{1}{2}\left(\frac{a}{2s}A+\frac{b}{2s}B+\frac{c}{2s}C\right)^2-\tfrac{1}{2}(\tfrac{1}{2}B+\tfrac{1}{2}C)^2$$

$$=\frac{1}{4s}(ac^2+ab^2+ba^2+ca^2)-\frac{1}{4s^2}(bca^2+cab^2+abc^2)-\tfrac{1}{4}a^2$$

$$OD^2=\frac{a(2b^2+2c^2-a^2+ab+ac-4bc)}{4(a+b+c)}$$

[3]. $GO^2=\frac{1}{3}(bc+ca+ab)-\dfrac{2abc}{a+b+c}-\frac{1}{9}(a^2+b^2+c^2)$

[4]. $GP^2=\frac{2}{9}(a^2+b^2+c^2)$

$$-\tfrac{1}{3}\cot A\cot B\cot C\{a^2(\tan A+3\cot A)$$
$$+b^2(\tan B+3\cot B)+c^2(\tan C+3\cot C)\}$$

Where P is the intersection of the perpendiculars of the triangle ABC. See 26.

150. *If* $$0=aA+bB+cC+\text{etc.} \tag{1}$$
then $$\frac{-\frac{1}{2}(bB+cC+\text{etc.})^2}{a}=\frac{-\frac{1}{2}(aA+cC+\text{etc.})^2}{b}=\text{etc.}$$
$$=(aA+bB+cC+\text{etc.})P^2$$
where P *is any point*

(1) gives $-aA=bB+cC+\text{etc.}$

multiply by P^2

therefore $a^2AP^2=(-a)(bBP^2+cCP^2+\text{etc.})$
$$-\tfrac{1}{2}(bB+cC+\text{etc.})^2$$
hence $-\frac{1}{2}(bB+cC+\text{etc.})^2=a(aA+bB+cC+\text{etc.})P^2$

similarly $-\frac{1}{2}(aA+cC+\text{etc.})^2=b(aA+bB+cC+\text{etc.})P^2$

and so on.

The truth of the proposition is now obvious.

Example. G being the centre of gravity of ABC, and P any point, since $0=A+B+C-3G$, therefore
$$-\tfrac{1}{2}(B+C-3G)^2=-\tfrac{1}{2}(A+C-3G)^2=-\tfrac{1}{2}(A+B-3G)^2$$
$$=\frac{-\frac{1}{2}(A+B+C)^2}{-3}$$
$$=AP^2+BP^2+CP^2-3GP^2=AG^2+BG^2+CG^2$$
or $$-BC^2+3BG^2+3GC^2=-AC^2+3GA^2+3GC^2$$
$$=-AB^2+3GA^2+3GB^2$$
$$=\tfrac{1}{3}(AB^2+AC^2+BC^2)=AP^2+BP^2+CP^2-3GP^2$$
$$=AG^2+BG^2+CG^2$$

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151. *If* A, B, C, D, *etc., be points on a circle, whose centre is* O *and radius* r, *and if*
$$0=aA+bB+cC+\text{etc.}$$
then
$$0=\tfrac{1}{2}(bB+cC+dD+\text{etc.})^2=\tfrac{1}{2}(aA+cC+\text{etc.})^2=\text{etc.}$$

For we have $\frac{\frac{1}{2}(bB+cC+\text{etc.})^2}{a}$

$$=\frac{\frac{1}{2}(aA+cC+\text{etc.})^2}{b}=\frac{\frac{1}{2}(aA+bB+dD+\text{etc.})^2}{c}$$

$=-(aA+bB+cC+\text{etc.})P^2$ where P is any point.

Let P coincide with O,

then $aAP^2+bBP^2+cCP^2+\text{etc.}=aAO^2+bBO^2+\text{etc.}$

$=(a+b+c+\text{etc.})r^2=0$ since $a+b+c+\text{etc.}=0$

therefore

$$\tfrac{1}{2}(bB+cC+\text{etc.})^2=0=\tfrac{1}{2}(aA+cC+\text{etc.})^2=\text{etc.}$$

Examples. [1]. ABCD is a quadrilateral inscribed in a circle, to prove AC.BD=CD.AB+AD.BC.

We have $0=BC.CD.DB.A-CD.DA.AC.B$
$+DA.AB.BD.C-AB.BC.CA.D$

hence

$0=(BC.CD.DB.A-CD.DA.AC.B+DA.AB.BD.C)^2$

or $0=-BC.CD.DB.CD.DA.AC.AB^2$
$+BC.CD.DB.DA.AB.BD.AC^2$
$-CD.DA.AC.DA.AB.BD.BC^2$

divide by BC.CD.DB.DA.AC.AB

therefore $0=-CD.AB+BD.AC-DA.BC.$

[2]. The parallelogram ABCD inscribed in a circle is a rectangle.

Here $0=A-B+C-D$ since ABCD is a parallelogram; but A, B, C, D being on a circle, this equation gives

$0=(A-B+C)^2$ or $0=-AB^2+AC^2-BC^2$

whence, by Euclid I. 48, the angle B is a right angle, and therefore the parallelogram is a rectangle.

Multiplication by Circle.

152. If through a fixed point F any straight line is drawn cutting a circle, centre O, radius r, the rectangle FP.FQ is constant, and equals $r^2-\mathrm{OF}^2$ when F is within the circle, and OF^2-r^2 when F is without the circle.

Definition. I shall take OF^2-r^2 *as the result of the multiplication of* F *by the circle, whose centre is* O *and radius* r. Hence the product of F by a circle is the square of the tangent from F to the circle when F is without it, but is minus the rectangle of the segments of any chord through F when F is within the circle. This product vanishes when F lies on the circumference.

153. *If* $(x+y+z+\text{etc.})\mathrm{P}=x\mathrm{A}+y\mathrm{B}+z\mathrm{C}+\text{etc.}$ (1) *and tangents real or imaginary,* p, a, b, c, etc., *be drawn from the points* P, A, B, C, etc., *to any circle,*

then $$(x+y+z+\text{etc.})^2p^2$$
$$=(x+y+z+\text{etc.})(xa^2+yb^2+zc^2+\text{etc.})$$
$$-\tfrac{1}{2}(a\mathrm{A}+b\mathrm{B}+c\mathrm{C}+\text{etc.})^2$$

Let O, r be the centre and radius of the circle, multiply (1) by O^2, then

$$(x+y+z+\text{etc.})^2\mathrm{PO}^2$$
$$=(x+y+z+\text{etc.})(x\mathrm{AO}^2+y\mathrm{BO}^2+z\mathrm{CO}^2+\text{etc.})$$
$$-\tfrac{1}{2}(a\mathrm{A}+b\mathrm{B}+c\mathrm{C}+\text{etc.})^2$$

but $(x+y+z+\text{etc.})^2r^2$
$$=(x+y+z+\text{etc.})(xr^2+yr^2+zr^2+\text{etc.})$$

Subtract this equation from the preceding, remark that $\mathrm{PO}^2-r^2=p^2$, $\mathrm{AO}^2-r^2=a^2$, and so on, and the required result is obtained.

Examples. [1]. If from the point C in the line AB a tangent CT be drawn to any circle, and from A, B, tangents AR, BS to the same, then

$$\mathrm{AB}.\mathrm{CT}^2=\mathrm{AC}.\mathrm{BS}^2+\mathrm{BC}.\mathrm{AR}^2-\mathrm{AC}.\mathrm{BC}.\mathrm{AB}.$$

For $$AB.C = AC.B + BC.A$$
$$\times \text{ circle.}$$

therefore

$$AB^2.CT^2 = AB.AC.BS^2 + AB.BC.AR^2 - AC.BC.AB^2$$

dividing by AB, the required result is obtained.

[2]. A chord PR of the circumscribed circle of the triangle ABC passes through its centre of gravity G, to prove $9GP.GR = a^2 + b^2 + c^2$.

$$3G = A + B + C$$
$$\times \text{ circumscribed circle}$$
$$9(-GP.GR) = 3(0+0+0) - (BC^2 + CA^2 + AB^2)$$

which is the required result.

Cor. Since

$$BC^2 + CA^2 + AB^2 = 9GP.GR = 9(q^2 - GQ^2) \quad \text{See 31.}$$

the sum of the squares on the sides of a triangle inscribed in a circle cannot be greater than 9 times the square of its radius.

[3]. The tangents at B, C to the circumscribed circle meet in D; to find their length.

$$(b^2 + c^2 - a^2)D = -a^2A + b^2B + c^2C \qquad \text{See 35.}$$
$$\times \text{ circumscribed circle}$$

therefore $(b^2 + c^2 - a^2)^2 DB^2$

$$= (-a^2 + b^2 + c^2)(-0 + 0 + 0) - (b^2c^2a^2 - c^2a^2b^2 - a^2b^2c^2)$$

hence $(b^2 + c^2 - a^2)DB = abc.$

[4]. To find the tangent from A to the nine points circle.

This circle passes through the middle points D, E, F of BC, CA, AB.

Also $$A = E - D + F$$
$$\times \text{ nine points circle}$$

then $(\text{required tangent})^2 = +ED^2 - EF^2 + DF^2$

$$= \tfrac{1}{4}(c^2 - a^2 + b^2) = \tfrac{1}{2}bc \cos A.$$

[5]. PGR being a chord of the inscribed circle through the centre of gravity G.

$$\text{GP.GR}=\frac{1}{6}(bc+ca+ab)-\frac{5}{36}(a^2+b^2+c^2)$$

This result is obtained by multiplying $3\text{G}=\text{A}+\text{B}+\text{C}$ by the inscribed circle.

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[6]. To prove that the nine points circle is touched internally by the inscribed circle.

$(a+b+c)\text{O}=a\text{A}+b\text{B}+c\text{C}$

$\times$ nine points circle, centre Q′, radius $=\frac{1}{2}q=$ $\frac{1}{2}$ radius of the circumscribed circle,

then, using Example 4,

$$(a+b+c)^2\left(\text{OQ}'^2-\frac{q^2}{4}\right)$$

$$=(a+b+c)\left(a\frac{bc\cos\text{A}}{2}+b\frac{ca\cos\text{B}}{2}+c\frac{ab\cos\text{C}}{2}\right)$$

$$-(abc^2+acb^2+bca^2)$$

$$(a+b+c)\left(\text{OQ}'^2-\frac{q^2}{4}\right)=\frac{abc}{2}(\cos\text{A}+\cos\text{B}+\cos\text{C}-2)$$

$$=\frac{abc}{2}\left(4\sin\frac{\text{A}}{2}\sin\frac{\text{B}}{2}\sin\frac{\text{C}}{2}-1\right)$$

or $$2s\left(\text{OQ}'^2-\frac{q^2}{4}\right)=\frac{abc}{2}\left(\frac{r}{q}-1\right)=\frac{4qsr}{2}\left(\frac{r}{q}-1\right)$$

$$\text{OQ}'^2=\frac{q^2}{4}+r^2-qr$$

$$\text{OQ}'=\frac{q}{2}-r=\text{difference of the radii}$$

or the circles touch internally.

[7]. The nine points circle is touched externally by each of the escribed circles.

[8]. If V, V_1, V_2, V_3 be the points of contact of the nine points circle with the inscribed and escribed circles, then VV_1 cuts BC at the foot of the bisector of the angle A; V_2V_3 cuts BC at the foot of the bisector of the exterior angle A; AV_1, BV_2, CV_3 intersect in one point V′ which lies on OQ′; and Q′V′OV is a line harmonically divided.

Since $$\left(\frac{q}{2}-r\right)V=\frac{q}{2}O-rQ'$$

or $$\left(\frac{1}{r}-\frac{2}{q}\right)V=\frac{O}{r}-\frac{2}{q}Q' \qquad (1)$$

and $$\left(\frac{1}{r_1}+\frac{2}{q}\right)V_1=\frac{O_1}{r_1}+\frac{2}{q}Q' \qquad (2)$$

$$\left(\frac{1}{r_2}+\frac{2}{q}\right)V_2=\frac{O_2}{r_2}+\frac{2}{q}Q' \qquad (3)$$

$$\left(\frac{1}{r_3}+\frac{2}{q}\right)V_3=\frac{O_3}{r_3}+\frac{2}{q}Q' \qquad (4)$$

First. (1)+(2) gives $\left(\frac{1}{r}-\frac{2}{q}\right)V+\left(\frac{1}{r_1}+\frac{2}{q}\right)V_1$

$$=\frac{O}{r}+\frac{O_1}{r_1}=\frac{aA+bB+cC}{2sr}+\frac{-aA+bB+cC}{2(s-a)r_1}$$

$$=\frac{bB+cC}{\text{area ABC}} \qquad (5)$$

or VV_1 meets BC in the point $bB+cC$, that is, in the foot of the bisector of the interior angle A.

Next. $$V_2\left(\frac{1}{r_2}+\frac{2}{q}\right)-V_3\left(\frac{1}{r_3}+\frac{2}{q}\right)=\frac{O_2}{r_2}-\frac{O_3}{r_3}$$

$$=\frac{aA-bB+cC}{2(s-b)r_2}-\frac{aA+bB-cC}{2(s-c)r_3}=\frac{cC-bB}{\text{area ABC}}$$

or V_2V_3 meets BC in the foot $cC-bB$ of the exterior angle A.

Lastly. From (5)

$$V_1\left(\frac{1}{r_1}+\frac{2}{q}\right)+\frac{aA}{\text{area ABC}}=V_2\left(\frac{1}{r_2}+\frac{2}{q}\right)+\frac{bB}{\text{area ABC}}$$

$$=V_3\left(\frac{1}{r_3}+\frac{2}{q}\right)+\frac{cC}{\text{area ABC}}=\frac{aA+bB+cC}{\text{area ABC}}-\left(\frac{1}{r}-\frac{2}{q}\right)V$$

$$=\frac{2O}{r}-\frac{O}{r}+\frac{2}{q}Q'$$

$$=\frac{O}{r}+\frac{2}{q}Q'$$

or V_1A, V_2B, V_3C meet in a point V', such that

$$\left(\frac{1}{r}+\frac{2}{q}\right)V'=\frac{O}{r}+\frac{2}{q}Q' \qquad (6)$$

from (1) and (6) $Q'V'OV$ are points in one right line which is harmonically divided.

[9]. ABCD is a quadrilateral circumscribing a circle; P, Q, R, S are the points of contact of AB, CD, BC, AD, to prove

$$PQ^2 : RS^2 = BC.AD : AB.CD.$$

Let E be the intersection of PQ, RS. Let a, b, c, d denote the lengths AP, BR, CQ, DS, then, see 104,

$$\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)E=\left(\frac{1}{a}+\frac{1}{b}\right)P+\left(\frac{1}{c}+\frac{1}{d}\right)Q$$

$$=\left(\frac{1}{a}+\frac{1}{d}\right)S+\left(\frac{1}{b}+\frac{1}{c}\right)R$$

$\times$ circle

therefore $\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)^2 PE.EQ=\left(\frac{1}{a}+\frac{1}{b}\right)\left(\frac{1}{c}+\frac{1}{d}\right)PQ^2$

$$=\left(\frac{1}{a}+\frac{1}{d}\right)\left(\frac{1}{b}+\frac{1}{c}\right)RS^2$$

therefore $(a+b)(c+d)PQ^2=(a+d)(b+c)RS^2$

or $\qquad AB.CD.PQ^2=AD.BC.RS^2.$

[10]. To prove that the sum of the squares on the

sides and diagonals of a polygon of n sides, inscribed in a circle, cannot be greater than n^2 times the square on the radius.

Let O be the centre, and r the radius of the circle, and let ABCD ... be the polygon, M its mean point, then $$nM = A + B + C + D + \text{etc.}$$

$$\times \text{ circle,}$$

$$n^2(OM^2 - r^2)$$
$$= -(AB^2 + AC^2 + AD^2 + \text{etc.} + BC^2 + \text{etc.} + CD^2 + \text{etc.})$$

hence

$$AB^2 + AC^2 + AD^2 + \text{etc.} + BC^2 + BD^2 + \text{etc.} + CD^2 + \text{etc.}$$
$$= n^2r^2 - n^2OM^2$$

and is never greater than n^2r^2.

Cor. When the polygon is regular, M coincides with O, and, therefore, the sum of the squares of the sides and diagonals of a regular polygon of n sides, is n^2 times the square of the radius of its circumscribing circle; and is greater than the sum of the squares on the sides and diagonals of any other polygon of the same number of sides inscribed in the same circle.

Multiplication by Circle P — Circle Q.

154. *If* $(a+b+c+\text{etc.})G = aA + bB + cC + \text{etc.}$ (1) *and* g', a', b', etc., *be tangents from* G, A, B, etc., *to a circle, and* g'', a'', b'', etc., *the tangents from the same points to another circle, then*

$$(a+b+c+\text{etc.})(g'^2 - g''^2)$$
$$= a(a'^2 - a''^2) + b(b'^2 - b''^2) + c(c'^2 - c''^2) + \text{etc.}$$

For let P, Q be the centres, and p, q the radii of the two circles. Multiplying (1) by $P^2 - Q^2$, therefore

$$(a+b+c+\text{etc.})(GP^2 - GQ^2)$$
$$= a(AP^2 - AQ^2) + b(BP^2 - BQ^2) + \text{etc.}$$

from each side of this equation subtract $(a+b+c+\text{etc}.p^2$, and to each of them add $(a+b+c+\text{etc})q^2$, therefore

$$(a+b+c+\text{etc.})\{(GP^2-p^2)-(GQ^2-q^2)\}$$
$$=a\{(AP^2-p^2)-(AQ^2-q^2)\}$$
$$+b\{(BP^2-p^2)-(BQ^2-q^2)\}+\text{etc.},$$

which is the required result, since $GP^2-p^2=g'^2$, and so on.

Examples. [1]. D, E is any line bisected by G, the centre of gravity of the triangle ABC; with centres D, E equal circles are described. To prove that the sum of the squares of the tangents to those circles from A, B, C are equal.

Let AA′, BB′, CC′ touch the circle D, and AA″, BB″, CC″ the circle E.

Since $3G=A+B+C$
$\times$ circle D—circle E.

Since the tangents from G to the two circles are equal, therefore

$$0=AA'^2-AA''^2+BB'^2-BB''^2+CC'^2-CC''^2.$$

[2]. The sum of the squares of the sides of a triangle ABC is equal to four times the sum of the squares of the tangents AA′, BB′, CC′ drawn from the vertices to the nine points circle.

Let D, E, F be the bisections of BC, CA, AB; then the circle about DEF is the nine points circle.

Since $2D=B+C,\ 2E=C+A,\ 2F=A+B$
therefore $D+E+F=A+B+C$
$\times$ circle about DEF—circle about ABC.

then $BD.DC+CE.EA+AF.FB=+AA'^2+BB'^2+CC'^2$

but $BD=DC=\dfrac{a}{2},\ CE=EA=\dfrac{b}{2},AF=FB=\dfrac{c}{2}$

hence $\dfrac{a^2}{4}+\dfrac{b^2}{4}+\dfrac{c^2}{4}=AA'^2+BB'^2+CC'^2.$

[3]. If the sides taken in order of the triangle ABC be divided in the same ratio in D, E, F, and a circle be described about DEF, and tangents AA′, BB′, CC′ drawn to it, then

$$AA'^2+BB'^2+CC'^2=BD\,.\,DC+CE\,.\,EA+AF\,.\,FB$$

By hypothesis

$$D=xB+(1-x)C,\ E=xC+(1-x)A,\ F=xA+(1-x)B$$

adding $\quad D+E+F=A+B+C$

$\times$ circle DEF—circle ABC

$$BD.DC+CE.EA+AF.FB=AA'^2+BB'^2+CC'^2.$$

[4]. The tangents real or imaginary to two circles from a point P in their common chord AB are equal.

Suppose P on BA produced, draw the tangents PQ, PT to the circles.

Since $\quad AB.P=BP.A-AP.B$

$\times$ circle ABQ—circle ABT

therefore $AB(PQ^2-PT^2)=BP(0-0)-AP(0-0)=0$
whence $PQ=PT$.

[5]. BAD, DAC are two circles; BAC a right line; BQ, CT are tangents to the circles DAC, BAD; to prove $BQ^2:CT^2=BA:CA$.

$$BC.A=BA.C+CA.B$$

$\times$ circle ADB—circle ADC

therefore $\quad 0=BA(CT^2-0)+CA(0-BQ^2)$
whence the required result.

Cor. It follows that if, through A, the intersection of two circles, a line BC terminated by them is so drawn that A bisects it, then the tangents from B, C to the circles C, B are equal.

[6]. BP is the common chord of two circles, AC a line terminated at the circumferences and bisected in B.

Join AP, CP meeting the circumferences again in A′, C′, and prove $PA.AA'=PC.CC'$.

$$2B=A+C$$

$$\times \text{ circle } ABP-\text{circle } BPC$$

$$0=(0-AP.AA')+(CP.CC'-0) \quad \text{whence, etc.}$$

[7]. If two circles, centres A, B, radii a, b, touch each other externally, the part x of their common tangent between the points C, D of contact is a mean proportional between the diameters.

Since AC is parallel to BD,

therefore $\qquad bA-bC=aB-aD$

multiply by circle B—circle A

therefore

$$b\{(a+b)^2-b^2+a^2\}-bx^2=a\{-b^2-(a+b)^2+a^2\}+ax^2$$

reducing $\qquad x^2=4ab.$

Multiplication by P^2—Circle.

155. *If* $(a+b+c+\text{etc.})G=aA+bB+cC+\text{etc.}$ (1) *and* g', a', b', c', etc., *denote the tangents from* G, A, B, C, etc. *to a circle, and* P *be any point, then*

$$(a+b+c+\text{etc.})(GP^2-g'^2)$$
$$=a(AP^2-a'^2)+b(BP^2-b'^2)+\text{etc.}$$

For let O be the centre of the circle, r its radius, then (1) multiplied by P^2-O^2 gives

$$(a+b+c+\text{etc.})(GP^2-GO^2)$$
$$=a(AP^2-AO^2)+b(BP^2-BO^2)+\text{etc.}$$

add $(a+b+c+\text{etc.})\ r^2$ to each side, then

$$(a+b+c+\text{etc.})\{GP^2-(GO^2-r^2)\}$$
$$=a\{AP^2-(AO^2-r^2)\}+b\{BP^2-(BO^2-r^2)\}+\text{etc.}$$

which is the required result, since $GO^2-r^2=g'^2$, and so on.

Examples. [1]. If a straight line, drawn from the vertex A of an isosceles triangle to meet the base BC in D, be produced to E on the circumference of the circle described about the triangle, then $DA.EA=CA^2$.

Here $\quad BC.D=BD.C+CD.B$

$\times A^2$—circle

therefore $BC(DA^2+DA.DE)=BD.CA^2+CD.BA^2$

but $\quad BA=CA$

hence $\quad DA^2+DA.DE=CA^2$, or $DA.AE=CA^2$.

[2]. ABCD being a quadrilateral inscribed in a circle, and P any point whatever,

$$\text{area } BCD.PA^2+\text{area } DAB.PC^2$$
$$=\text{area } CDA.PB^2+\text{area } ABC.PD^2$$

For ABCD being a quadrilateral,

$$0=\text{area } BCD.A-\text{area } CDA.B$$
$$+\text{area } DAB.C-\text{area } ABC.D$$

$\times P^2$—the circle

therefore $0=\text{area } BCD.AP^2-\text{area } CDA.BP^2$

$$+\text{area } DAB.CP^2-\text{area } ABC.DP^2.$$

[3]. ABCD is a parallelogram, AFGH is a circle cutting AB, AC, AD in F, G, H, then

$$AB.AF+AD.AH=AC.AG.$$

For ABCD being a parallelogram

$$0=A-B+C-D$$

$\times A^2$—circle

therefore $0=-(BA^2-BA.BF)$

$$+(CA^2-CA.CG)-(DA^2-DA.DH)$$

or $\quad 0=-BA.FA+CA.GA-DA.HA.$

Circles through one Point.

156. *If* $(x+y+z+\text{etc.})G=xA+yB+zC+\text{etc.}$ (1) *and if* P, Q *be any two points; and circles, centres* G, A, B, etc. *be described, all passing through* Q; *and tangents real or imaginary be drawn from* P *to these circles, then* g, a, b, c, *etc. denoting these tangents,*

$$(x+y+z+\text{etc.})g^2=xa^2+yb^2+zc^2+\text{etc.}$$

For multiply (1) by $P^2—Q^2$

therefore $(x+y+z+\text{etc.})(GP^2—GQ^2)$

$$=x(AP^2—AQ^2)+y(BP^2—BQ^2)+\text{etc.}$$

but $GP^2—GQ^2=g^2$, $AP^2—AQ^2=a^2$, etc.

hence the required result.

Example. If, with the angular points of a parallelogram as centres, circles be described all passing through one point Q, and PA′, PB′, PC′, PD′ be tangents to these circles from any point P, then

$$PA'^2+PC'^2=PB'^2+PD'^2$$

For ABCD being a parallelogram,

$$0=A—B+C—D$$
$$\times P^2—Q^2$$

therefore $0=(AP^2—AQ^2)—(BP^2—BQ^2)$

$$+(CP^2—CQ^2)—(DP^2—DQ^2)$$

or $0=A'P^2—B'P^2+C'P^2—D'P^2.$

CHAPTER IX.

PERPENDICULARITY.

157. *If* R *be a point in the straight line* AB, *bisecting at right angles the right line* PQ, *then* $R(P^2-Q^2)=0$.

For if AB, PQ intersect in C, by hypothesis $PC=CQ$; and the angles PCR, QCR are equal, each being a right angle; also CR is common to the two triangles PCR, QCR; which are equal in every respect by Euclid, Book I., prop. 4. Hence $RP=RQ$, therefore $RP^2=RQ^2$, and $R(P^2-Q^2)=0$.

158. Conversely: *If* $R(P^2-Q^2)=0$, *then* R *lies in the straight line, bisecting* PQ *at right angles.*

From $R(P^2-Q^2)$, we derive successively,

$$RP^2-RQ^2=0, RP=RQ.$$

Bisect PQ in C, join CR, then the three sides PC, CR, RP of the triangle PCR, being equal to the three sides QC, CR, RQ of QCR, the two triangles are equal in every respect by Euclid, Book I., prop. 8. Hence the adjacent angles PCR, QCR are equal, and therefore are right angles; and since $PC=CQ$, R lies in the line bisecting PQ at right angles.

Examples. [1]. The three lines bisecting at right angles the sides of the triangle ABC meet in a point.

Let the lines bisecting AB, AC at right angles meet in D, then

$$D(A^2-B^2)=0 \text{ and } D(C^2-A^2)=0$$

adding and reducing, therefore

$$D(C^2-B^2)=0$$

which means that D lies in the line bisecting CB at right angles. See 158.

[2]. If D lies on AB, and is equally distant from A, B, C, the angle ACB is a right angle.

Here $2D=A+B$

multiply by A^2-C^2

therefore $2D(A^2-C^2)=AB^2-AC^2-BC^2$

but by hypothesis $D(A^2-C^2)=0$

therefore $AC^2+BC^2=AB^2$, and the angle ACB a right angle.

159. *If* $R(P^2-Q^2)=0$ (1)

and if $(x+y+z+\text{etc.})R=xA+yB+zC+\text{etc.}$ (2)

then $(xA+yB+zC+\text{etc.})(P^2-Q^2)=0$ (3)

For multiplying (2) by P^2-Q^2, we get

$$(x+y+z+\text{etc.})R(P^2-Q^2)$$
$$=(xA+yB+zC+\text{etc.})(P^2-Q^2)$$

but from (1) $(x+y+z+\text{etc.})R(P^2-Q^2)=0$, hence the required result.

160. Conversely:

If $(xA+yB+zC+\text{etc.})(P^2-Q^2)=0$,

then the point $(x+y+z+\text{etc.})R=xA+yB+zC+\text{etc.}$, *lies in the line bisecting* PQ *at right angles.*

Examples. [1]. To prove

$$3(GA^2-GB^2)=-PA^2+PB^2$$

where P is the intersection of the perpendiculars of ABC.

For $Q(A^2-B^2)=0$, Q being the centre of the circle about ABC

but $3G=2Q+P$. See 32.

therefore $(3G+P)(A^2-B^2)=0$, whence the required result.

[2]. To prove $OA^2+O_1A^2+O_2A^2+O_3A^2$

$$=OB^2+O_1B^2+O_2B^2+O_3B^2$$
$$=OC^2+O_1C^2+O_2C^2+O_3C^2$$
$$=OP^2+O_1P^2+O_2P^2+O_3P^2$$

where P is any point on the circumscribed circle.

Since $Q(A^2-P^2)=0$

and $4Q=O+O_1+O_2+O_3$ See 54.

therefore $(O+O_1+O_2+O_3)(A^2-P^2)=0$

or $OA^2+O_1A^2+O_2A^2+O_3A^2=OP^2+O_1P^2+O_2P^2+O_3P^2$.

[3]. If P be any point on the circle circumscribing the triangle ABC, then

$PA^2 \sin 2A+PB^2 \sin 2B+PC^2 \sin 2C=4$ area of ABC.

For $Q(P^2-A^2)=0$

but $(\ldots\ldots)Q=\sin 2A.A+\sin 2B.B+\sin 2C.C$

therefore $(\sin 2A.A+\sin 2B.B+\sin 2C.C)(P^2-A^2)=0$

then $\sin 2A.PA^2+\sin 2B.PB^2+\sin 2C.PC^2$

$$=\sin 2B.c^2+\sin 2C.b^2=4 \text{ area} ABC$$

Corollary. If ABC be an equilateral triangle and P any point on its circumscribed circle,

$$PA^2+PB^2+PC^2=2BC^2.$$

[4]. In the triangle ABC,

$$\sin 2A(b^2-c^2)+a^2(\sin 2B-\sin 2C)=0$$

For $Q(B^2-C^2)=0$

whence $(\sin 2A.A+\sin 2B.B+\sin 2C.C)(B^2-C^2)=0$

or $\sin 2A(c^2-b^2)-\sin 2B.a^2+\sin 2C.a^2=0$.

161. *If* A, B *be two points equally distant from* C, D *then*
$$(xA+yB)(C^2-D^2)=0$$

Let E be the point $(x+y)E=xA+yB$ then E lies on AB, and therefore is equally distant from C, D. Hence $E(C^2-D^2)=0$, and the required result. See Art. 159.

Example.—Let D, E, F be the points of contact of the circle inscribed in the triangle ABC, then
$$BE^2-BF^2 : CF^2-CE^2=c : b$$

For the centre O of the inscribed circle and the vertex A being each equally distant from E, F, therefore
$$\{(a+b+c)O-aA\}\{E^2-F^2\}=0$$
or
$$(bB+cC)(E^2-F^2)=0$$
therefore
$$b(BE^2-BF^2)=c(CF^2-CE^2).$$

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162. *If the line* AB *is perpendicular to the line* CD, *then*
$$0=(A-B)(C^2-D^2)$$
or
$$0=(C-D)(A^2-B^2)$$
and
$$0=(pA-pB)(C^2-D^2)$$
or
$$0=(pC-pD)(A^2-B^2)$$ *where p is any quantity.*

Let AB cut CD in E, then the angles at E being right angles,
$$AC^2=AE^2+CE^2$$
$$AD^2=AE^2+DE^2$$
subtract
$$A(C^2-D^2)=(C-D)E^2$$
similarly
$$B(C^2-D^2)=(C-D)E^2$$
whence
$$(A-B)(C^2-D^2)=0$$
p times this result gives
$$(pA-pB)(C^2-D^2)=0.$$

163. Conversely. *If* $(A-B)(C^2-D^2)=0$ *then* AB *is perpendicular on* CD.

Proof, when AB produced meets CD itself in E; let the angle AED be denoted by θ, then

$$AC^2=CE^2+AE^2+2CE.AE\cos\theta$$
$$AD^2=DE^2+AE^2-2DE.AE\cos\theta$$

therefore $AC^2-AD^2=CE^2-DE^2+2AE\cos\theta(CE+DE)$
similarly $BC^2-BD^2=CE^2-ED^2+2BE\cos\theta(CE+DE)$
but by hypothesis $AC^2-AD^2=BC^2-BD^2$, therefore

$$AE\cos\theta=BE\cos\theta, \text{ or } AB\cos\theta=0$$

but AB is not zero, therefore $\cos\theta=0$, or θ is a right angle.

Examples. [1]. If P, Q be two points each equally distant from A and B, then PQ is perpendicular on AB.

Here $\quad P(A^2-B^2)=0$
and $\quad Q(A^2-B^2)=0$

therefore subtracting $(P-Q)(A^2-B^2)=0$
which means that PQ is perpendicular on AB. See 163.

[2]. The common chord PQ of two circles is perpendicular to the line joining their centres A, B.

For since the radius AP=the radius AQ, therefore

$$A(P^2-Q^2)=0$$
similarly $\quad B(P^2-Q^2)=0$

subtracting, $(A-B)(P^2-Q^2)=0$, or AB, PQ are at right angles. See 163.

[3]. In the triangle ABC, if CD is perpendicular on AB, then $DA^2-DB^2=CA^2-CB^2$.

CD being perpendicular on AB,
therefore $(C-D)(A^2-B^2)=0$
or $CA^2-CB^2=DA^2-DB^2$

[4]. The perpendiculars of a triangle ABC meet in a point.

Let the perpendiculars BE, CF from B, C on CA, AB meet in P, then

$$(P-B)(C^2-A^2)=0$$

and $$(P-C)(A^2-B^2)=0$$

or $$P(C^2-A^2)=BC^2-AB^2$$

and $$P(A^2-B^2)=CA^2-BC^2$$

add $$P(C^2-B^2)=A(C^2-B^2)$$

or $(P-A)(C^2-B^2)=0$ or PA is perpendicular on CB.

[5]. Upon the sides BC, CA, AB of the triangle ABC, or upon these produced, let fall the perpendiculars DE, DF, DG, from the point D, within or without the triangle, then $BE^2+CF^2+AG^2=CE^2+BG^2+AF^2$.

For $$(D-E)(B^2-C^2)=0$$

and $$(D-F)(C^2-A^2)=0$$

and $$(D-G)(A^2-B^2)=0$$

eliminate D, which is done by adding, then

$E(B^2-C^2)+F(C^2-A^2)+G(A^2-B^2)=0$, whence, etc.

[6]. In any triangle ABC, if AE be drawn from the angle A in any direction, and BE, CF be perpendiculars on it from BC; and EG, FG be drawn to the point of bisection G of BC, then shall GE=GF.

For $$(B-E)(E^2-F^2)=0$$

and $$(C-F)(E^2-F^2)=0$$

add $$(B+C-E-F)(E^2-F^2)=0$$

or $(B+C)(E^2-F^2)=(E+F)(E^2-F^2)=-EF^2+FE^2=0$

or $$2G(E^2-F^2)=0, \text{ whence } GE=GF.$$

[7]. If the diagonals of a parallelogram ABCD be at right angles to each other, the parallelogram is equilateral.

Here $$0=A-B+C-D \qquad (1)$$

and $$(A-C)(B^2-D^2)=0 \qquad (2)$$

now from (1) $$A-C=B-2C+D$$

multiply by $\qquad B^2-D^2$
hence using (2) $\quad 0=-2CB^2+DB^2-DB^2+2CD^2$,
whence $CB=CD$.

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164. *If* $\quad (P-Q)(R^2-S^2)=0 \qquad (1)$
and $\quad (x+y+z+\text{etc.})P=xA+yB+zC+\text{etc.} \qquad (2)$
and $\quad (a+b+c+\text{etc.})Q=aA+bB+cC+\text{etc.} \qquad (3)$
then

$$\left(\frac{xA+yB+zC+\text{etc.}}{x+y+z+\text{etc.}}-\frac{aA+bB+cC+\text{etc.}}{a+b+c+\text{etc.}}\right)(R^2-S^2)=0 \quad (4)$$

For (2) multiplied by R^2-S^2 gives

$$(x+y+z+\text{etc.})P(R^2-S^2)=(xA+yB+zC+\text{etc.})(R^2-S^2)$$

whence $\quad P(R^2-S^2)=\dfrac{xA+yB+zC+\text{etc.}}{x+y+z+\text{etc.}}(R^2-S^2)$

similarly $\quad Q(R^2-S^2)=\dfrac{aA+bB+cC+\text{etc.}}{a+b+c+\text{etc.}}(R^2-S^2)$

subtracting and using equation (1) we get the required result.

Again, equation (4) is equivalent to

$$(R-S)\left(\frac{xA^2+yB^2+zC^2+\text{etc}}{x+y+z+\text{etc.}}-\frac{aA^2+bB^2+cC^2+\text{etc}}{a+b+c+\text{etc.}}\right)=0$$

Hence, as above, if

$$(p+q+r+\text{etc.})R=pA+qB+rC+\text{etc.}$$

and $\quad (h+k+l+\text{etc.})S=hA+kB+lC+\text{etc.}$

then
$$\left\{\left(\frac{pA+qB+rC+\text{etc.}}{p+q+r+\text{etc.}}-\frac{hA+kB+lC+\text{etc.}}{h+k+l+\text{etc.}}\right)\left(\frac{xA^2+yB^2+\text{etc.}}{x+y+\text{etc.}}-\frac{aA^2+bB^2+\text{etc.}}{a+b+\text{etc.}}\right)\right\}=0$$

or *in equation* $0=(P-Q)(R^2-S^2)$ *we may replace* P, Q, R, S, *each by an equivalent system of points.*

Examples. [1]. G being the centre of gravity of the triangle ABC, and GK perpendicular on BC, AK is produced to V, so that $KV=\frac{1}{2}AK$; to prove that V is equally distant from B and C.

Here $\quad (G-K)(B^2-C^2)=0$

therefore $\quad (3G-3K)(B^2-C^2)=0$

but $\quad 3G=A+B+C$ and $3K=A+2V$

therefore $\quad (A+B+C-A-2V)(B^2-C^2)=0$

$$2V(B^2-C^2)=(B+C)(B^2-C^2)=-BC^2+CB^2=0$$

or V is equally distant from B and C. See 158.

[2]. If from the middle point D of BC, one of the sides of a right-angled triangle ABC, a perpendicular DE be drawn on the hypothenuse AB, the difference of the squares of the segments into which it is divided is equal to the square of the other side.

Here $\quad (D-E)(A^2-B^2)=0$

therefore $\quad (2D-2E)(A^2-B^2)=0$

or $\quad (C+B-2E)(A^2-B^2)=0$

therefore $2(EA^2-EB^2)=CA^2-CB^2+BA^2=2CA^2$ since ACB is a right angle.

[3]. If $2AP^2+BP^2+CP^2=AB^2+AC^2$ the locus of P is the circle, on the bisector AD of the side BC as diameter.

For the above equation gives

$$(2P-B-C)(A^2-P^2)=0$$

or $(2P-2D)(A^2-P^2)=0$ or $(P-D)(A^2-P^2)=0$ or PD, PA are at right angles, whence P lies on the circle on AD as diameter.

[4]. If a straight line BD be drawn from one of the acute angles of a right-angled triangle ABC, bisecting the opposite side, the square upon that line is less than the square on the hypothenuse AB, by three times the square upon half the side bisected.

Here $$2D=A+C$$
and $$(A-C)(C^2-B^2)=0$$
Eliminate C, then $(2A-2D)(2D^2-A^2-B^2)=0$
hence $$BD^2=AB^2-3AD^2.$$

[5]. BAC is an isosceles triangle, right angled at A; E is the point of trisection of BC nearer to C, F the middle point of AC; to prove that AE, BF are at right angles.

AE, BF are at right angles
if $(A-E)(B^2-F^2)=0$, if $(3A-3E)(2B^2-2F^2)=0$
if $(3A-2C-B)(2B^2-A^2-C^2)=0$, which is true.

[6]. If ABC be a triangle right angled at C, and G be its centre of gravity, then

$$GA^2-GC^2=\frac{1}{3}AC^2$$

and $$GA^2+GB^2=\frac{5}{9}AB^2.$$

Here $$3G=A+B+C \qquad (1)$$
and $$(C-B)(A^2-C^2)=0 \qquad (2)$$

The elimination of B gives the first result, that of C the second.

[7]. The line OQ is perpendicular to the line which passes through the points of intersection of the sides of the triangle ABC, and of the bisectors of the opposite external angles. (London University, 1865.)

Let D, E be the feet of the bisectors of the external angles at A, B; we must prove

$$(Q-O)(D^2-E^2)=0$$

or $$\{(a+b+c)Q-aA-bB-cC\}\{(c-a)(bB^2-cC^2)$$
$$-(b-c)(cC^2-aA^2)\}=0$$

or reducing and remarking that $QA^2=QB^2=QC^2$, we must have

$$(a\mathrm{A}+b\mathrm{B}+c\mathrm{C})\{(bc-ab)\mathrm{B}^2+(ab-ac)\mathrm{A}^2+(ac-bc)\mathrm{C}^2\}=0$$

or $0=(abc-a^2b)c^2+(a^2c-abc)b^2+(ab^2-abc)c^2$

$$+(abc-b^2c)a^2+(bc^2-abc)a^2+(abc-ac^2)b^2$$

which is true.

[8]. If the bisectors of the angles B, C meet the opposite sides in E, F, then EF is perpendicular on QO_1.

[9]. To prove

$$a(\mathrm{OA}^2-\mathrm{OB}^2)-c(\mathrm{OC}^2-\mathrm{OB}^2)=ac(c-a)$$

this result is derivable by the elimination of O_1 from the equations $\quad(\mathrm{O}-\mathrm{B})(\mathrm{B}^2-\mathrm{O}_1{}^2)=0$

$$(-a+b+c)\mathrm{O}_1=-a\mathrm{A}+b\mathrm{B}+c\mathrm{C}.$$

165. *If* $\quad(\mathrm{A}-\mathrm{B})(\mathrm{C}^2-\mathrm{D}^2)=0 \qquad (1)$

then also $\quad\left(\dfrac{a\mathrm{A}+b\mathrm{B}}{a+b}-\dfrac{x\mathrm{A}+y\mathrm{B}}{x+y}\right)(\mathrm{C}^2-\mathrm{D}^2)=0 \qquad (2)$

For (1) means that AB is perpendicular on CD; hence CD is also perpendicular on the line joining any two points on AB.

Now $\dfrac{a\mathrm{A}+b\mathrm{B}}{a+b}$ and $\dfrac{x\mathrm{A}+y\mathrm{B}}{x+y}$ are two points on AB; hence the truth of the equation (2).

Hence also: *If* $(\mathrm{A}-\mathrm{B})(\mathrm{C}^2-\mathrm{D}^2)=0$ *then*

$$\left(\frac{a\mathrm{A}+b\mathrm{B}}{a+b}-\frac{x\mathrm{A}+y\mathrm{B}}{x+y}\right)\left(\frac{p\mathrm{C}^2+q\mathrm{D}^2}{p+q}-\frac{r\mathrm{C}^2+s\mathrm{D}^2}{r+s}\right)=0.$$

166. *If* $(a+b+c+\text{etc.})\mathrm{P}=a\mathrm{A}+b\mathrm{B}+c\mathrm{C}+\text{etc.}$ (1) *and* PQ *be perpendicular on* LM, *then*

$$(a+b+c+\text{etc.})\mathrm{Q}(\mathrm{L}^2-\mathrm{M}^2)$$
$$=(a\mathrm{A}+b\mathrm{B}+c\mathrm{C}+\text{etc.})(\mathrm{L}^2-\mathrm{M}^2)$$

For (1) multiplied by $\mathrm{L}^2-\mathrm{M}^2$ gives

$$(a+b+c+\text{etc.})\mathrm{P}(\mathrm{L}^2-\mathrm{M}^2)$$
$$=(a\mathrm{A}+b\mathrm{B}+c\mathrm{C}+\text{etc.})(\mathrm{L}^2-\mathrm{M}^2)$$

but $\qquad(\mathrm{Q}-\mathrm{P})(\mathrm{L}^2-\mathrm{M}^2)=0$ by the hypothesis

therefore $\quad P(L^2-M^2)=Q(L^2-M^2)$

whence the required result.

Example. To prove $(a+b)(CO_1^2-CO_2^2)$

$$=a(AO_1^2-AO_2^2)+b(BO_1^2-BO_2^2)$$

Since $\quad (a+b+c)O=aA+bB+cC$

and since OC is perpendicular on $O_1\,O_2$

therefore

$$(a+b+c)C(O_1^2-O_2^2)=(aA+bB+cC)(O_1^2-O_2^2)$$

whence $(a+b)C(O_1^2-O_2^2)=(aA+bB)(O_1^2-O_2^2)$.

167. *If* $\quad (C-D)(A^2-B^2)=0$

and $\quad (E-F)(A^2-B^2)=0$

then CD, EF *being both perpendicular on* AB, *therefore* CD, EF *are parallel, and* $\dfrac{C-D}{CD}=\dfrac{E-F}{EF}$

Again: *if* $\quad (C-D)(A^2-B^2)=0$

and $\quad (G-H)(E^2-F^2)=0$

and $\quad \dfrac{A-B}{AB}=\dfrac{E-F}{EF}$ *then* $\dfrac{C-D}{CD}=\dfrac{G-H}{GH}$

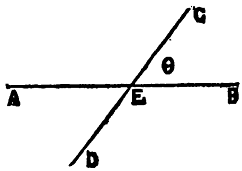
For AB being parallel to EF, therefore CD, GH, perpendiculars on the parallel lines AB, EF, are themselves parallel.

CHAPTER X.

ANGLES.

168. *If the directions* AB, DC *make the angle* θ, *then*

$$\cos\theta = \frac{(C-D)(A^2-B^2)}{2CD.AB.}$$



Let AB, DC intersect in E

then $CA^2 = AE^2 + EC^2 + 2AE.EC \cos\theta$

and $CB^2 = CE^2 + EB^2 - 2CE.EB \cos\theta$

whence $C(A^2-B^2) = AE^2 - EB^2 + 2CE.AB \cos\theta$ (1)

again $DA^2 = AE^2 + DE^2 - 2AE.DE \cos\theta$

and $DB^2 = DE^2 + EB^2 + 2DE.EB \cos\theta$

whence $D(A^2-B^2) = AE^2 - EB^2 - 2DE.AB \cos\theta$ (2)

(1)—(2) gives $(C-D)(A^2-B^2) = 2AB.CD \cos\theta$.

Examples. [1]. ABC is a triangle whose base AB is divided in E and produced to F, so that AE : EB=AC : BC and AF : FB=AC : BC. Join CE, CF, and show that the angle ECF is a right angle.

By question $(a+b)E=aA+bB$

and $(a-b)F=aA-bB$

also $2EC.CF \cos ECF=(E-C)(C^2-F^2)$

but $(E-C)(C^2-F^2)=\left(\frac{aA+bB}{a+b}-C\right)\left(C^2-\frac{aA^2-bB^2}{a-b}\right)$

$$=\frac{\{aA+bB-(a+b)C\}\{(a-b)C^2-aA^2+bB^2\}}{a^2-b^2}=0$$

therefore ECF is a right angle.

[2]. The squares of the diagonals of a trapezium ABCD are together equal to the squares of its two oblique sides AD, BC, with twice the rectangle contained by its parallel sides.

Since the angle between the directions AB, DC is zero, therefore

$$(C-D)(A^2-B^2)=2CD.AB \cos 0$$

or $CA^2+BD^2-CB^2-DA^2=2CD.AB$, whence, etc.

[3]. The diagonals of an equilateral quadrilateral ABCD intersect at right angles.

Let E be the intersection, then AEB is the angle between the directions CA, DB, therefore

$$(B-D)(C^2-A^2)=2BD.CA \cos AEB$$

but $(B-D)(C^2-A^2)=BC^2+DA^2-DC^2-BA^2=0$ since the sides of ABCD are equal. Hence $\cos AEB=0$, $AEB=$a right angle.

---

169. *If θ be the angle between the directions* AB, DC, *then*

$$(C-D)(A^2-B^2)=4 \cot \theta \,.\, \text{area } (C-D)AB$$

As in 168, $(C-D)(A^2-B^2)=2CD.AB \cos \theta$. (1)

Now if CD produced meets AB itself in E

$$2 \text{ area } ABC = AB.CE \sin\theta$$

and $$2 \text{ area } ABD = AB.ED \sin\theta$$

subtracting $$2 \text{ area } AB\,(C-D) = AB.CD \sin\theta \quad (2)$$

From (1) and (2) the required result is readily obtained.

But if CD is cut in E by AB we shall have $2 \text{ area } ABC + 2 \text{ area } ABD = AB.CD \sin\theta$

or still $$2 \text{ area } AB(C-D) = AB.CD \sin\theta$$

provided we pay attention to the convention of signs given in the chapter on areas.

Therefore $(C-D)(A^2-B^2) = 4\cot\theta \text{ area } (C-D)AB.$

Examples. [1]. To find the cotangent of the angle A of the triangle ABC.

A is the angle between the directions AB, AC, therefore $(C-A)(A^2-B^2) = 4 \cot A \text{ area } (C-A)AB$

or $$b^2-a^2+c^2 = 4 \cot A \text{ area } ABC.$$

[2]. To find $\cot\frac{A}{2}$ in the triangle ABC.

Draw AD, the bisector of the angle A, then $\frac{A}{2}$ is the angle between the directions AC, AD,

therefore $$(D-A)(A^2-C^2) = 4 \cot\frac{A}{2} \text{ area } (D-A)AC$$

or $$\left(\frac{bB+cC}{b+c}-A\right)(A^2-C^2) = 4 \cot\frac{A}{2} \text{ area} \frac{bB+cC}{b+c}AC$$

or $$\{bB+cC-(b+c)A\}(A^2-C^2) = 4\cot\frac{A}{2}\, b \text{ area } BAC$$

or $$bc^2-ba^2+cb^2+(b+c)b^2 = 4b\cot\frac{A}{2} \text{ area } BAC.$$

Hence $$\cot\frac{A}{2} = \left(\frac{s(s-a)}{(s-b)(s-c)}\right)^{\frac{1}{2}}$$

[3]. CD is the bisector of the side AB of the triangle ABC, to find the cotangent of the angle CDB.

This angle is the inclination of the directions AB, DC, therefore $(C-D)(A^2-B^2)=4\cot CDB \text{ area } (C-D)AB$.

Therefore

$$(2C-A-B)(A^2-B^2)=4\cot CDB \text{ area } (2C-A-B)AB$$

$$2b^2-2a^2+c^2-c^2=4\cot CDB.2 \text{ area } CAB$$

$$\cot CDB=\frac{b^2-a^2}{4 \text{ area } ABC}.$$

[4]. If a line be drawn from the vertex C of the triangle ABC to the point of the trisection D of the base nearer to A, to prove

$$4\cot DCA \,.\, \text{area } ABC=5b^2+a^2-c^2.$$

DCA is the angle between the directions CD, CA, therefore

$$(D-C)(C^2-A^2)=4\cot DCA \text{ area } (D-C)CA$$

but $3D=2A+B$,

hence $(2A+B-3C)(C^2-A^2)$

$$=4\cot DCA \text{ area } (2A+B-3C)CA$$

or $2b^2+a^2-c^2+3b^2=4\cot DCA \text{ area } BCA$, hence, etc.

[5]. To find the cotangent of the angle EAD between the lines AD, AE, drawn from the vertex A of the triangle ABC to trisect the base.

Let B, D, E, C be the order of the points on BC, then EAD is the angle between the directions AE, AD, therefore

$$(E-A)(A^2-D^2)=4\cot EAD \text{ area } (E-A)AD$$

but $3E=2C+B$ and $3D=2B+C$

hence $(2C+B-3A)(3A^2-2B^2-C^2)$

$$=4\cot EAD \text{ area } (2C+B-3A)A(2B+C)$$

or $6b^2-4a^2+3c^2-a^2+6c^2+3b^2$

$$=4\cot EAD(2CA+BA)(2B+C)$$

$$9b^2+9c^2-5a^2=4\cot EAD\ (4CAB+BAC)$$

$$=12\cot EAD \text{ area } ABC.$$

Corollary. 1. If A be a right angle

$$\cot EAD=\frac{2}{3}\left(\frac{b}{c}+\frac{c}{b}\right)$$

Corollary. 2. If ABC be an equilateral triangle

$$\cos EAD=\frac{13}{14}.$$

[6]. To find the angle which the side BC of the triangle ABC subtends at the centre of gravity G.

CGB is the angle between the directions GC, GB, therefore

$$(C-G)(G^2-B^2)=4\cot CGB \text{ area } (C-G)GB$$

but $$3G=A+B+C$$

therefore

$$(2C-A-B)(A^2+C^2-2B^2)=36\cot CGB\,.\,\text{area } CGB$$

or $$b^2+c^2-5a^2=12\cot CGB\,.\,\text{area } ABC.$$

Corollary. When A is a right angle

$$\cot CGB=-\frac{2}{3}\left(\frac{b}{c}+\frac{c}{b}\right)$$

From this and Corollary 1 of the preceding example, we obtain the following theorem:—

ABC is a triangle right angled in A; D, E points on the hypothenuse, such that BD=DE=EC; AD is joined, intersecting in M the bisector of the side AB; AE intersects in N the bisector of the side AC, and the two bisectors intersect in G; then AMGN is a quadrilateral, which may be inscribed in a circle.

We also deduce, from this last theorem, that, when ABC is an isosceles triangle right angled in A, then AE drawn to the point of trisection of BC, nearer to C, is perpendicular to the bisector of the side AC.

For then the angles ANG, AMG become equal.

[7]. If CD bisects AB, and CE be perpendicular on AB, then

$$\cot \mathrm{DCE} = \frac{2 \sin \mathrm{A} \sin \mathrm{B}}{\sin(\mathrm{A}-\mathrm{B})}.$$

[8]. D being the point of contact of BC and the inscribed circle, to find the angle ADC.

This angle is the inclination of the directions DA, BC, therefore

$$\begin{aligned}(\mathrm{A}-\mathrm{D})(\mathrm{B}^2-\mathrm{C}^2) &= 4 \cot \mathrm{ADC} \text{ area } (\mathrm{A}-\mathrm{D})\mathrm{BC} \\ &= 4 \cot \mathrm{ADC} \,.\, \text{area ABC}\end{aligned}$$

but $$a\mathrm{D} = (s-b)\mathrm{C} + (s-c)\mathrm{B};$$

therefore after reductions

$$(c-b)(c+b-a) = 4 \cot \mathrm{ADC} \text{ area ABC}.$$

[9]. $$\mathrm{Cot}\, \mathrm{OGO}_1 = \frac{a^4-b^4-c^4+4\, b^2c^2+b^3c+bc^3-5a^2bc}{12a(c-b) \text{ area ABC.}}$$

[10]. $$\mathrm{Cot}\, \mathrm{O}_2\mathrm{GO}_3 = \frac{b^4+c^4-a^4-4b^2c^2+b^3c+bc^3-5a^2bc}{12a(b+c) \text{ area ABC.}}$$

[11]. To find the angle between the diagonals of the rectangle ABCD.

Let E be the intersection of the diagonals. Then CEB being the angle between the directions AC, DB, we have

$$(\mathrm{C}-\mathrm{A})(\mathrm{D}^2-\mathrm{B}^2) = 4 \cot \mathrm{CEB} \,.\, \text{area } (\mathrm{C}-\mathrm{A})\mathrm{DB}$$

whence $$\cot \mathrm{CEB} = \frac{\mathrm{AB}^2-\mathrm{BC}^2}{2\mathrm{AB}.\mathrm{BC}}.$$

[12]. To find the angle between the diagonals of a quadrilateral ABCD.

Let E be the intersection of the diagonals, then, as the angle CEB is the inclination of the directions AC, DB, therefore

$$(C-A)(D^2-B^2)=4 \cot CEB \text{ area } (C-A)DB$$

or $$CD^2+AB^2-CB^2-AD^2$$
$$=4 \cot CEB\{\text{area } CDB+\text{area } ADB\}$$

or $$\cot CEB=\frac{CD^2+AB^2-CB^2-AD^2}{4 \text{ area quadrilateral } ABCD}$$

Corollary. The diagonals of ABCD are at right angles if the sums of the squares of the opposite sides are equal.

[13]. To find the angle between two opposite sides AB, CD of a quadrilateral ABCD.

Let θ be that angle, then θ being the angle between the directions BA, CD, we have

$$(A-B)(C^2-D^2)=4 \cot \theta \text{ area } (A-B)CD$$

or $$AC^2+BD^2-BC^2-AD^2$$
$$=4 \cot \theta\ (\text{area } ACD-\text{area } BCD)$$

Corollary. Hence, if the sum of the squares of the diagonals of a quadrilateral be equal to the sum of the squares of two opposite sides, the remaining two sides are at right angles.

[14]. To find the angle between the lines bisecting the opposite sides of a quadrilateral.

Let E, F, G, H be the bisections of AB, BC, CD, DA, and let EG, FH intersect in K, then GKF is the inclination of EG, HF, and therefore

$$(G-E)(H^2-F^2)=4 \cot GKF \text{ area } (G-E)HF$$

or $$(C+D-A-B)(D^2+A^2-C^2-B^2)$$
$$=16 \cot GKF \text{ area } EFGH$$

$$AC^2-DB^2=4 \cot GKF \text{ area } ABCD.$$

Corollary. Hence, if the diagonals of a quadrilateral

are equal, the lines bisecting opposite sides are at right angles.

[15]. To find the angle between the diagonals of a quadrilateral ABCD circumscribing a circle.

Let a, b, c, d be the tangents from A, B, C, D to the circle; let E be the intersection of the diagonals; then the angle AED is the inclination of the directions CA, BD; therefore

$$(A-C)(B^2-D^2)=4 \cot AED \text{ area } (A-C)BD$$

therefore

$$AB^2+CD^2-BC^2-AD^2=4 \cot AED \text{ area } ABCD$$

but $AB=a+b$, $BC=b+c$, $CD=c+d$, $DA=d+a$

hence $2 \cot AED \, . \text{area } ABCD=(a-c)(b-d)$.

Corollary. When $a=c$, then also the angles A, C are equal, and therefore

If two opposite angles of a quadrilateral circumscribing a circle be equal, its diagonals intersect at right angles.

The diagonal joining the equal angles is bisected by the other, which passes through the centre of the circle.

CHAPTER XI.

TRIANGULAR OR AREAL EQUATION OF A STRAIGHT LINE.

170. *Definition of the triangular or areal co-ordinates of a point.*

Let A, B, C be fixed points, called the points of reference; and let $(x+y+z)P=xA+yB+zC$, then P is a point in the plane ABC, and if we know x, y, z, or more accurately, the ratios $x:y:z$, the position of P is perfectly defined. The quantities x, y, z, or quantities proportional to x, y, z, are called the co-ordinates of P, with regard to the triangle of reference ABC; and since

$$x:y:z=\text{area of the triangle PBC}:\text{PCA}:\text{PAB}$$

x, y, z are also called the triangular or areal co-ordinates of P.

171. *Definition of the areal equation of a line straight or curved.*

As in ordinary analytical geometry, the triangular or areal equation of a line straight or curved is the equation between the areal co-ordinates of *any* point on that line, that equation containing no other variable.

172. *To find the areal equation of a straight line* LM.

Let AA′, BB′, CC′ be parallel lines meeting LM in A′, B′, C′. Let P be any point on LM, let x, y, z be the co-ordinates of P, so that

$$(x+y+z)\mathrm{P}=x\mathrm{A}+y\mathrm{B}+z\mathrm{C} \qquad (1)$$

it is required to find an equation between x, y, z, and no other variable.

This is readily done, by taking distances in (1) from LM parallel to AA′, then

$$0=x\mathrm{AA'}+y\mathrm{BB'}+z\mathrm{CC'} \qquad (2)$$

the required equation. AA′ being taken as positive, BB′ will be positive, if on the same side of LM as AA′, negative if on opposite side, and the same with regard to CC′.

Examples. [1]. The areal equation to BC is $0=x$.

For the perpendiculars from A, B, C on BC, are proportional to 1, 0, 0, and therefore the equation to BC is

$$0=x.1+y.0+z.0, \text{ or } 0=x.$$

Similarly the equation to CA is $0=y$

to AB is $0=z$.

[2]. The areal equation of AD perpendicular on BC is $0=y \cot \mathrm{B}-z \cot c$.

For the perpendiculars from A, B, C on AD are 0, $c \cos \mathrm{B}$, $-b \cos \mathrm{C}$, and therefore proportional to 0, cot B, $-\cot \mathrm{C}$; hence the equation to AD is

$$0=x.0+y \cot \mathrm{B}+z(-\cot \mathrm{C}), \text{ or etc.}$$

Similarly the equation of BE perpendicular on CA is

$$0=z \cot \mathrm{C}-x \cot \mathrm{A}$$

that of CF perpendicular on AB is $0=x \cot \mathrm{A}-y \cot \mathrm{B}$.

[3]. The areal equation of EF bisecting in E, F the sides CA, AB, is $0=-x+y+z$.

For the perpendiculars from A,B,C on EF are of equal length; but that from A being on opposite side of EF

with regard to those from B, C, these perpendiculars are proportional to -1, 1, 1, and the equation of EF is

$$0=-x+y+z.$$

[4]. To find the equation of the bisector AD of the side BC.

Let $(x+y+z)P=xA+yB+zC$ be any point on AD. Take distances from AD parallel to BC, then

$$0=0+y\text{BD}+z(-\text{CD})$$

but BD$=$CD, therefore the required equation is $0=y-z$.

[5]. To find the equation of the bisector AD of the internal angle A of the triangle of reference ABC.

Let $(x+y+z)P=xA+yB+zC$ be any point on AD. Take distances from AD parallel to BC, then

$$0=y\text{BD}-z\text{CD}$$

but BD : CD$=$AB : AC$=c:b$. The required equation is

$$0=\frac{y}{b}-\frac{z}{c}.$$

[6]. The equation of the bisector of the exterior angle A of the triangle of reference ABC is $0=\frac{y}{b}+\frac{z}{c}$.

[7]. To find the areal equation of DE bisecting BC at right angles in D.

Let $(x+y+z)P=xA+yB+zC$ be any point on DE. Draw AF perpendicular on BC, then if BFDC be the order of the points on BC, taking distances from DE

$$0=x\text{FD}+y\text{BD}-z\text{CD} \quad\text{or}\quad 0=x\left(b\cos C-\frac{a}{2}\right)+y\,\frac{a}{2}-z\,\frac{a}{2}$$

and the required equation is $0=x(b^2-c^2)+(y-z)a^2$.

173. *The general equation of the first degree in x, y, x represents a right line.*

This general equation is $0=lx+my+nz$ (1)
where l, m, n are constants. To each value of the ratio

$y:x$, (1) only gives one definite value of the ratio $z:x$, and therefore but one definite point. The assemblage of all the points thus obtained from (1), by giving to $y:x$ all possible values, is the locus of the equation (1).

Let (a, b, c), (a', b', c'), (a'', b'', c'') be the co-ordinates of *any* three points P, Q, R on the locus of (1)

then $$0=la+mb+nc$$
$$0=la'+mb'+nc'$$
$$0=la''+mb''+nc''$$

whence eliminating l, m, n

$$0=a(b'c''-c'b'')+b(c'a''-a'c'')+c(a\,b''-b'a'') \qquad (2)$$

But $$(a+b+c)\mathrm{P}=a\mathrm{A}+b\mathrm{B}+c\mathrm{C}$$
$$(a'+b'+c')\mathrm{Q}=a'\mathrm{A}+b'\mathrm{B}+c'\mathrm{C}$$
$$(a''+b''+c'')\mathrm{R}=a''\mathrm{A}+b''\mathrm{B}+c''\mathrm{C},$$ hence, see 139,

$(a+b+c)(a'+b'+c')(a''+b''+c'')$ area PQR

$$=\{a(b'c''-c'b'')+b(c'a''-a'c'')+c(a'b''-b'a'')\}\,\mathrm{ABC}$$
$$=0 \quad \text{See (2)}$$

Hence any three points P, Q, R on the locus of (1) lie in one right line; therefore the locus itself is necessarily a right line.

174. *Intersections of lines.*

Being given the equations to two lines, if we make them simultaneous, x, y, z mean the co-ordinates of a point which is on both lines. Therefore the co-ordinates of the intersection, or intersections of two lines, are found by solving their equations as simultaneous equations.

Again, by combining the equations to two lines, we obtain the equation to a third line, which passes through the intersections of the first two.

175. *To find the intersections* D, E, F *of the right line* $0=lx+my+nz$ *with the sides of the triangle of reference.*

At D, $x=0$, and D lying on $0=lx+my+nz$, therefore at D

$$0=my+nz \quad \text{or} \quad y:z=\frac{1}{m}:-\frac{1}{n}$$

hence $\left(\frac{1}{m}-\frac{1}{n}\right)\mathrm{D}=\frac{\mathrm{B}}{m}-\frac{\mathrm{C}}{n}$

similarly $\left(\frac{1}{n}-\frac{1}{l}\right)\mathrm{E}=\frac{\mathrm{C}}{n}-\frac{\mathrm{A}}{l}$

and $\left(\frac{1}{l}-\frac{1}{m}\right)\mathrm{F}=\frac{\mathrm{A}}{l}-\frac{\mathrm{B}}{m}$

176. Areal equation of the line joining two given points D, E.

Let $(x+y+z)\mathrm{P}=x\mathrm{A}+y\mathrm{B}+z\mathrm{C}$ be any point on DE.

Let $(p+q+r)\mathrm{D}=p\mathrm{A}+q\mathrm{B}+r\mathrm{C}$

$(d+e+f)\mathrm{E}=d\mathrm{A}+e\mathrm{B}+f\mathrm{C}$

then $(x+y+z)(p+q+r)(d+e+f)$ area PDE

$$=\begin{vmatrix} x, & y, & z \\ p, & q, & r \\ d, & e, & f \end{vmatrix} \text{area ABC.}$$ See 139.

but area PDE=0, since P, D, E are in one right line, therefore

$$0=\begin{vmatrix} x, & y, & z \\ p, & q, & r \\ d, & e, & f \end{vmatrix}=x(qf-re)+y(rd-pf)+z(pe-dq)$$

is the required equation.

177. *Other solution.* Let $(x+y+z)\mathrm{P}=x\mathrm{A}+y\mathrm{B}+z\mathrm{C}$ be any point on DE, multiply by DE, then since area PDE=0, as P, D, E are in one right line, we get for the required equation

$$0=x \text{ area ADE}+y \text{ area BDE}+z \text{ area CDE}$$

attention being paid to the convention of signs respecting areas.

Examples. [1]. E, F being the middle points of CA, AB, to find the equation to EF.

The co-ordinates of E are (1, 0, 1), those of F (1, 1, 0), therefore the equation to EF is

$$0=\begin{vmatrix} x, & y, & z \\ 1, & 0, & 1 \\ 1, & 1, & 0 \end{vmatrix}=-x+y+z,$$

and this equation means that P being any point on EF,

area PBC=area PCA+area PAB.

[2]. The equation to QO is

$$0=\text{cosec } A(\cos B-\cos C)x+\text{cosec } B(\cos C-\cos A)y$$
$$+\text{cosec } C(\cos A-\cos B)z.$$

[3]. Areal equation of the line joining E, F, the feet of the bisectors of the angles B, C of the triangle of reference ABC.

The co-ordinates of E are $(a, 0, c)$, those of F are $(a, b, 0)$, and therefore the equation to EF is

$$0=-\frac{x}{a}+\frac{y}{b}+\frac{z}{c}$$

Cor. This line cuts BC in the point bB—cC, see 175, which is the foot of the bisector of the external angle A.

[4]. Areal equation of the line joining D, E, the feet of the bisectors of the external angles A, B of the triangle of reference ABC.

The co-ordinates of D are $(0, b, -c)$, those of E are $(a, 0, -c)$; hence the equation to DE is

$$0=\frac{x}{a}+\frac{y}{b}+\frac{z}{c}$$

(i.) This line cuts AB in the point aA—bB, see 175, which is the foot F of the bisector of the exterior angle

C. Therefore the feet of the bisectors of the exterior angles of a triangle lie on one right line.

(ii.) Let P be any point on DF, then the above equation means

$$0=\frac{\text{area PBC}}{\text{BC}}-\frac{\text{area PCA}}{\text{CA}}+\frac{\text{area PAB}}{\text{AB}}$$

hence the perpendicular from P on AC is equal to the sum of the perpendiculars from P on BC, AB.

[5]. The equation to GO is

$$0=(b-c)x+(c-a)y+(a-b)z$$

The line GO meets BC in the point D, such that

$$\begin{aligned}(b+c-2a)\text{D}&=(b-a)\text{B}+(c-a)\text{C}\\&=(b\text{B}+c\text{C})-a(\text{B}+\text{C}) \qquad (1)\end{aligned}$$

Similarly the areal equation to GO_1 is

$$0=(b-c)x+(c+a)y-(a+b)z$$

and GO_1 meets BC in D_1, such that

$$\begin{aligned}(b+c+2a)\text{D}_1&=(b+a)\text{B}+(c+a)\text{C}\\&=(b\text{B}+c\text{C})+a(\text{B}+\text{C}) \qquad (2)\end{aligned}$$

From (1) and (2) we see that D_1 and D divide harmonically the line joining the foot of the bisector of interior angle A with the middle point of BC.

Let GO_2 meet CA in E_2, let GO_3 meet AB in F_3

then $\quad (c+a+2b)\text{E}_2=(c+b)\text{C}+(a+b)\text{A} \qquad (3)$

and $\quad (a+b+2c)\text{F}_3=(a+c)\text{A}+(b+c)\text{B} \qquad (4)$

From (2), (3), (4) we readily derive that AD_1, BE_2, CF_3 meet in the point K, such that

$$(\ldots\ldots)\text{K}=\frac{\text{A}}{b+c}+\frac{\text{B}}{c+a}+\frac{\text{C}}{a+b}$$

K and G′, see 57, are reciprocal points. Easy reductions give

$$(\ldots\ldots)\text{K}=(a^2\text{A}+b^2\text{B}+c^2\text{C})+(bc+ca+ab)(\text{A}+\text{B}+\text{C}).$$

Now $a^2\text{A}+b^2\text{B}+c^2\text{C}$ is the common intersection T of the lines joining each vertex of ABC with the inter-

section of the tangents to the circumscribed circle at the other two vertices. Therefore K lies on TG and

$$TK : 3KG = bc+ca+ab : a^2+b^2+c^2.$$

[6]. If ABC be a given triangle, P any given point; and AD the fourth harmonic to AB, AP, AC intersect BC in D; and BE the fourth harmonic to BC, BP, BA intersect CA in E; and CF the fourth harmonic of CA, CP, CB intersect AB in F; then D, E, F lie on the right line $0=\frac{x}{p}+\frac{y}{q}+\frac{z}{r}$, where p, q, r are the co-ordinates of P. (Ferrers, Trilinear Co-ordinates, p. 28.)

Produce AP to meet BC in D′,

since $\qquad (p+q+r)P=pA+qB+rC,$

therefore $(p+q+r)P-pA=qB+rC=(q+r)D'$. See 13.

But BD′CD is a line harmonically divided.

hence $\qquad (q-r)D=qB-rC$

similarly $\quad (r-p)E=rC-pA$

and $\qquad (p-q)F=pA-qB$

The equation of DE, the line joining the points $D(0, q, -r)$, $E(-p, 0, r)$, is

$$0=\frac{x}{p}+\frac{y}{q}+\frac{z}{r}$$

This line meets AB in the point $pA-qB$, see 175, that is in the point F.

178. *The areal equation of the line* HK *bisecting at right angles the line* DE *is*

$$0=x(AD^2-AE^2)+y(BD^2-BE^2)+z(CD^2-CE^2).$$

Let $(x+y+z)P=xA+yB+zC$ be any point on HK, multiply by D^2-E^2, hence the above result, since PD^2-PE^2.

Example.—To find the areal equation to the line bisecting BC at right angles.

Let $(x+y+z)P=xA+yB+zC$ be any point on the required line, multiply by B^2-C^2, then since $PB^2=PC^2$, we get $0=x(AB^2-AC^2)+y(-BC^2)+zCB^2$

or $0=x(b^2-c^2)+ya^2-za^2$ (1) *Result*

Similarly $0=y(c^2-a^2)+zb^2-xb^2$ (2)

and $0=z(a^2-b^2)+xc^2-yc^2$ (3)

are the equations to the lines bisecting CA, AB at right angles. Since the equation (3) is derivable from (1) and (2), it follows that *the three lines bisecting at right angles the sides of a triangle meet in one point.* See 174.

179. *To find the areal equation of a line through the given point* F, *perpendicular to the line* DE.

Let $(x+y+z)P=xA+yB+zC$ be any point on the required line,

multiply by D^2-E^2,

therefore $(x+y+z)(PD^2-PE^2)$

$$=x(AD^2-AE^2)+y(BD^2-BE^2)+z(CD^2-CE^2)$$

But PF being perpendicular on DE

$$(P-F)(D^2-E^2)=0$$

therefore $PD^2-PE^2=FD^2-FE^2$

hence the required equation is

$$(x+y+z)(FD^2-FE^2)$$
$$=x(AD^2-AE^2)+y(BD^2-BE^2)+z(CD^2-CE^2)$$

Example. To find the equation of the perpendicular from A on BC.

Let $(x+y+z)P=xA+yB+zC$ be any point on this perpendicular,

multiply by B^2-C^2, therefore

$$(x+y+z)(PB^2-PC^2)=x(c^2-b^2)-ya^2+za^2$$

but AP being perpendicular on BC

$$(P-A)(B^2-C^2)=0$$

therefore $PB^2—PC^2=AB^2—AC^2=c^2—b^2$
and the required equation is

$$(x+y+z)(c^2—b^2)=x(c^2—b^2)—ya^2+za^2$$

or $$0=y(c^2+a^2—b^2)—z(a^2+b^2—c^2)$$

180. *To find the areal equation of a line* PF *through the given point* F *parallel to the line* DE.

Let $(x+y+z)P=xA+yB+zC$ be any point on the required line, $\times DE$
therefore $(x+y+z)$ area PDE

$$=x \text{ area ADE}+y \text{ area BDE}+z \text{ area CDE}$$

but PF being parallel to DE, area PDE=area FDE
therefore the required equation is

$$(x+y+z) \text{ area FDE}=x\text{ADE}+y\text{BDE}+z\text{CDE}.$$

Example. To find the equation to the line through the centre O of the inscribed circle parallel to BC.

Let $(x+y+z)P=xA+yB+zC$ be any point on the line.

Multiply by BC,
therefore $$(x+y+z)PBC=xABC$$
but area PBC=area OBC, since OP, BC are parallel;
therefore the required equation is

$$(x+y+z)OBC=xABC$$

or $$0=(b+c)x—a(y+z).$$

---

181. *If the co-efficients l, m, n of $0=lx+my+nz$ be subjected to a condition of the first degree, this right line passes through a fixed point.*

Let $0=lp+mq+nr$ be the given condition, then evidently the point (p, q, r) lies on the line

$$0=lx+my+nz.$$

182. Conversely: *If a right line always passes through a fixed point, there is a relation of the first degree between its co-efficients.*

183. *To find the co-ordinates of the intersection of the two lines* $0=lx+my+nz$ *and* $0=l'x+m'y+n'z$.

At the intersection, and at the intersection only, the two equations are simultaneous; and then x, y, z are the co-ordinates of the required point.

Solving the two equations, we get

$$x:y:z=mn'-m'n:nl'-n'l:lm'-l'm.$$

Or the intersection is the point

$$(mn'-m'n)\mathrm{A}+(nl'-n'l)\mathrm{B}+(lm'-l'm)\mathrm{C}.$$

184. *The lines* $0=lx+my+nz$, $0=l'x+m'y+n'z$ *are parallel, if* $0=mn'-m'n+n'l-n'l+lm'-l'm$.

For these lines are parallel, when their intersection

$(mn'-m'n)\mathrm{A}+(nl'-n'l)\mathrm{B}+(lm'-l'm)\mathrm{C}$ (See 183)

lies at infinity; that is, see 15, when

$$0=mn'-m'n+nl'-n'l+lm'-l'm.$$

Examples. [1]. The line EF bisecting the sides AC, AB of the triangle ABC is parallel to the base BC.

The equation to EF is $\quad 0=-1x+1y+1z$

that of AC is $\quad 0=1x+0y+0z$

These lines are parallel, for $0=0+1-1$.

[2]. The lines $\quad 0=lx+my+nz$ and

$(x+y+z)p=lx+my+nz$ are parallel.

For $m(n-p)-n(m-p)+n(l-p)-l(n-p)$
$+l(m-p)-m(l-p)=0$

185. *To find the condition that the three right lines* $0=lx+my+nz$, $\quad 0=l'x+m'y+n'z$, $\quad 0=l''x+m''y+n''z$ *shall pass through one point.*

The three lines will pass through one point, if their equations can be simultaneous, that is,

$$\text{if}\quad 0=\begin{vmatrix} l, & m, & n \\ l', & m', & n' \\ l'', & m'', & n'' \end{vmatrix} = l(m'n''-n'm'')+m(n'l''-l'n'') \\ +n(l'm''-m'l'')$$

We may remark that this condition is the same as that necessary in order that the three points (l, m, n), (l', m', n'), (l'', m'', n'') shall lie in one right line.

Example. The bisectors of the sides of a triangle ABC meet in one point.

These bisectors are $0=y-z$, $0=z-x$, $0=x-y$, and they meet in one point if

$$0=\begin{vmatrix} 0, & 1, & -1 \\ -1, & 0, & 1 \\ 1, & -1, & 0 \end{vmatrix} \text{ or if } 0=0+1-1.$$

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186. *To find the magnitude of the resultant of the forces* lBC *acting along* BC, mCA *along* CA, nAB *along* AB.

Let R denote this resultant, and θ its inclination to BC, then resolving along BC and perpendicular to BC, we get

$$\begin{aligned} \text{R} \cos \theta &= l\text{BC} - m\text{CA} \cos \text{C} - n\text{AB} \cos \text{B} \\ &= la - mb \cos \text{C} - nc \cos \text{B} \end{aligned}$$

$$\text{R} \sin \theta = mc\text{A} \sin \text{C} - n\text{AB} \sin \text{B} = mb \sin \text{C} - nc \sin \text{B}$$

square and add, therefore

$$\text{R}^2 = l^2a^2 + m^2b^2 + n^2c^2 - 2mnbc \cos \text{A} - 2nlca \cos \text{B} \\ -2mlab \cos \text{C}.$$

187. *Notation.* For shortness sake, I shall denote by $D(l, m, n)$ this expression, thus

$$D(l, m, n) = l^2a^2 + m^2b^2 + n^2c^2$$
$$-2mnbc \cos A - 2nlca \cos B - 2lmab \cos C$$
$$= a^2(l-m)(l-n) + b^2(m-n)(m-l) + c^2(n-l)(n-m).$$

188. *This resultant acts in the line* $0 = lx + my + nz$.

For take moments about any point P in the resultant, then

moment of resultant $= 2l$ area $PBC + 2m\ PCA + 2n\ PAB$ but P being on the resultant, the moment of the resultant vanishes, therefore

$$0 = l \text{ area } PBC + mPCA + nPAB$$

or if x, y, z be the areal co-ordinates of P, any point on the resultant, $0 = lx + my + nz$ which is the equation to the line of action of the resultant.

189. *To find the length of the perpendicular from the point* $P(p, q, r)$ *on the line* $0 = lx + my + nz$.

The forces lBC along BC, mCA along CA, nAB along AB, have a resultant $D^{\frac{1}{2}}(l, m, n)$ acting along

$$0 = lx + my + nz.$$

Take moments about P, then

$$D^{\frac{1}{2}}(l, m, n) \,.\, \text{required perpendicular}$$
$$= lBC \frac{p}{p+q+r} \frac{2 \text{ area } ABC}{BC}$$
$$+ mCA \frac{q}{p+q+r} \frac{2 \text{ area } ABC}{CA} + nAB \frac{r}{p+q+r} \frac{2 \text{ area } ABC}{AB}$$

(see Chapter IV.), therefore

$$\text{the required perpendicular} = \frac{2 \text{ area } ABC(lp + mq + nr)}{(p+q+r)D^{\frac{1}{2}}(l, m, n)}$$

190. *To find the distance between the parallel lines*

$$0 = lx + my + nz \qquad (1)$$

and $0 = (l-p)x + (m-p)y + (n-p)z. \qquad (2)$ See 184.

Let (e, f, g) be any point on (2), then the required distance is the perpendicular from (e, f, g) on (1).

It is
$$\frac{2(le+mf+ng) \text{ area ABC}}{(e+f+g)\mathrm{D}^{\frac{1}{2}}(l, m, n)}$$
but (e, f, g) lying on (2)
$$le+mf+ng=p(e+f+g)$$
hence the required distance is
$$\frac{2p \text{ area ABC}}{\mathrm{D}^{\frac{1}{2}}(l, m, n)}$$

191. *Through a given point (e, f, g) to draw a line parallel to the line $0=lx+my+nz$.* (1)

Let $\quad 0=(l-p)x+(m-p)y+(n-p)z \qquad (2)$

be the required line, then, see 184, it is parallel to (1), and it passes through (e, f, g) if
$$0=(l-p)e+(m-p)f+(n-p)g$$
whence $\quad (e+f+g)p=le+mf+ng$

and the required equation is
$$(le+mf+ng)(x+y+z)=(e+f+g)(lx+my+nz)$$

192. *The line $0=x+y+z$ lies at infinity. It is called the line at infinity.*

For the perpendicular on it from any point (p, q, r) at a finite distance from the triangle of reference ABC (so that $p+q+r$ is not zero) is
$$\frac{2(p+q+r)\text{area ABC}}{(p+q+r)\mathrm{D}^{\frac{1}{2}}(1, 1, 1)} \text{ or } \frac{2 \text{ area ABC}}{\mathrm{D}^{\frac{1}{2}}(1, 1, 1)}$$
but $\mathrm{D}^{\frac{1}{2}}(1, 1, 1)=0$. See 187.

Therefore the perpendicular is infinite, and the line $0=x+y+z$ being at an infinite distance from the triangle ABC lies at infinity.

193. *To find the angle between the lines*
$$0=lx+my+nz \ (1), \quad 0=l'x+m'y+n'z \ (2)$$

The forces lBC along BC, mCA along CA, nAB along AB have a resultant $D^{\frac{1}{2}}(l, m, n)$ acting along (1). Similarly the forces l'BC along BC, m'CA along CA, n'AB have a resultant $D^{\frac{1}{2}}(l', m', n')$ acting along (2).

Hence if θ be the angle between (1) and (2) the square of the resultant of the forces $(l+l')$BC along BC, $(m+m')$CA along CA, $(n+n')$AB along AB is

$D(l, m, n)+D(l', m', n')+2D^{\frac{1}{2}}(l, m, n)D^{\frac{1}{2}}(l', m', n') \cos \theta.$

But the square of this resultant is also

$$D(l+l', m+m', n+n')$$

therefore $D(l+l', m+m', n+n')=D(l, m, n)+D(l', m', n')$
$$+2D^{\frac{1}{2}}(l, m, n)D^{\frac{1}{2}}(l', m', n') \cos \theta$$

whence

$$\cos \theta=\frac{D(l+l', m+m', n+n')-D(l, m, n)-D(l', m', n')}{2D^{\frac{1}{2}}(l, m, n)D^{\frac{1}{2}}(l', m', n')}$$

194. *To find the condition of perpendicularity of the lines* $0=lx+my+nz$, $0=l'x+m'y+n'z$.

When these lines are at right angles, the cosine of the angle between them vanishes, that is,

$$D(l+l', m+m', n+n')=D(l, m, n)+D(l', m', n')$$

or
$$(l+l')^2a^2+(m+m')^2b^2+(n+n')^2c^2$$
$$-2(m+m')(n+n')\,bc \cos A$$
$$-2(n+n')(l+l')\,ca \cos B$$
$$-2(l+l')(m+m')ab \cos C$$
$$=l^2a^2+m^2b^2+n^2c^2-2mn\,bc \cos A-2nl\,ca \cos B$$
$$-2lm\,ab \cos C+l'^2a^2+m'^2b^2+n'^2c^2-2m'n'bc \cos A$$
$$-2n'l'ca \cos B-2l'm'ab \cos C.$$

Whence, after reductions, the required condition is

$$0=ll'a^2+mm'b^2+nn'c^2-(mn'+m'n)bc \cos A$$
$$-(nl'+n'l)ca \cos B-(lm'+l'm)ab \cos C.$$

This condition may also be written

$$0=l'\frac{d\mathrm{D}(l, m, n)}{dl}+m'\frac{d\mathrm{D}}{dm}+n'\frac{d\mathrm{D}}{dn}$$

the differential co-efficients being partial differential co-efficients.

It may also be written

$$0=\{(m-n)\mathrm{A}+(n-l)\mathrm{B}+(l-m)\mathrm{C}\}\{(m'-n')\mathrm{A}^2 \\ +(n'-l')\mathrm{B}^2+(l'-m')\mathrm{C}^2\}.$$

195. *The line at infinity is perpendicular on any line whatever.*

Let $0=lx+my+nz$ be any line whatever, the line at infinity is $0=x+y+z$, and the two lines are at right angles, since one of the two factors of the last expression in 194 is at once zero.

196. *Through a given point (p, q, r) to draw a perpendicular on the given line $0=lx+my+nz$.*

Let $0=l'x+m'y+n'z$ (1) be the required line then $0=l'p+m'q+n'r$ (2) since it passes through the point (p, q, r),

also $$0=l'\frac{d\mathrm{D}(l, m, n)}{dl}+m'\frac{d\mathrm{D}}{dm}+n'\frac{d\mathrm{D}}{dn} \qquad (3)$$

since it is perpendicular on $0=lx+my+nz$.

Eliminating l', m', n' between (1), (2), (3), the required equation is

$$0=x\left\{q\frac{d\mathrm{D}}{dn}-r\frac{d\mathrm{D}}{dm}\right\}+y\left\{r\frac{d\mathrm{D}}{dl}-p\frac{d\mathrm{D}}{dn}\right\} \\ +z\left\{p\frac{d\mathrm{D}}{dm}-q\frac{d\mathrm{D}}{dl}\right\}$$

---

197. *To find the distance between the points*

$$(p+q+r)\mathrm{D}=p\mathrm{A}+q\mathrm{B}+r\mathrm{C}$$
$$(e+f+g)\mathrm{E}=e\mathrm{A}+f\mathrm{B}+g\mathrm{C}$$

Multiplying these equations, we get as a true equation of forces

$$(p+q+r)(e+f+g)\mathrm{DE}$$
$$=(qg-rf)\mathrm{BC}+(re-pg)\mathrm{CA}+(pf-eq)\mathrm{AB}$$

that is, the force $(p+q+r)(e+f+g)\mathrm{DE}$ acting along DE is the resultant of the forces $(qg-rf)\mathrm{BC}$ along BC, $(re-pg)\mathrm{CA}$ along CA, $(pf-eq)\mathrm{AB}$ along AB.

Hence equating values of the magnitude of this resultant

$$(p+q+r)(e+f+g)\mathrm{DE}=\mathrm{D}^{\frac{1}{2}}\{qg-fr,\ re-pg,\ pf-eq\}.$$

Whence $$\mathrm{DE}=\frac{\mathrm{D}^{\frac{1}{2}}(qg-fr,\ re-pg,\ pf-eq)}{(p+q+r)(e+f+g)}$$

198. *To find the perpendicular from* A(1, 0, 0) *on the line* OQ *whose equation is*

$$0=x(\sin 2\mathrm{B}\sin \mathrm{C}-\sin 2\mathrm{C}\sin \mathrm{B})+y \text{ etc.}+z \text{ etc.}$$

The required perpendicular is

$$\frac{2(\sin 2\mathrm{B}\sin \mathrm{C}-\sin 2\mathrm{C}\sin \mathrm{B})\text{area ABC}}{\mathrm{D}^{\frac{1}{2}}(\sin 2\mathrm{B}\sin \mathrm{C}-\sin 2\mathrm{C}\sin \mathrm{B}, \text{ etc., etc.})} \quad \text{See 189}$$

$$=\frac{4\sin \mathrm{B}\sin \mathrm{C}(\cos \mathrm{B}-\cos \mathrm{C})\text{ area ABC}}{(\sin 2\mathrm{A}+\sin 2\mathrm{B}+\sin 2\mathrm{C})(\sin \mathrm{A}+\sin \mathrm{B}+\sin \mathrm{C})\mathrm{QO}}$$

See 197

$$=\frac{(\cos \mathrm{B}-\cos \mathrm{C})\text{ area ABC}}{4\sin \mathrm{A}\cos\frac{\mathrm{A}}{2}\cos\frac{\mathrm{B}}{2}\cos\frac{\mathrm{C}}{2}\sqrt{q^2-2qr}}$$

where q, r are the radii of the circumscribed and inscribed circles.

199. The reader will easily prove the following facts:—

(i.) $D(l, m, n)$ is the sum of the squares of $mb\sin \mathrm{C}-nc\sin \mathrm{B}$ and $la-mb\cos \mathrm{C}-nc\cos \mathrm{B}$.

(ii.) If $D(l, m, n)=0$, then $l=m=n$.

(iii.) $D(x\pm l, y\pm l, z\pm l)=D(x, y, z)$.

(iv.) $\dfrac{dD}{dl}+\dfrac{dD}{dm}+\dfrac{dD}{dn}=0.$

(v.) If e, f, g be the perpendiculars from A, B, C upon any straight line, then $D^{\frac{1}{2}}(e, f, g)=2$ area ABC, or

$$a^2(e-f)(e-g)+b^2(f-g)(f-e)+c^2(g-e)(g-f) =(2 \text{ area ABC})^2.$$

For instance, let the straight line be one of the perpendiculars of the triangle ABC, then the following formula involving the interior angles of that triangle is derived,

$$1=\cos^2 A+\cos^2 B+\cos^2 C+2\cos A\cos B\cos C.$$

Again, if with the angular points A, B of the triangle ABC as centres, and with radii α, β, circles be described, and if γ, γ' denote the perpendiculars from C on the external common tangents of these circles, then

$$\gamma(c\gamma-2b\cos A\beta-2a\cos B\alpha)$$
$$=\gamma'(c\gamma'-2b\cos A\beta-2a\cos B\alpha).$$

Hence, if C lie on one external common tangent, and γ be the perpendicular from C on the other external common tangent, then

$$c\gamma=2b\cos A\beta+2a\cos B\alpha.$$

Rectilinear Loci.

200. *If* $\{a+b+c+\text{etc.}+x(a'+b'+c'+\text{etc.})\}P$
$=(a+xa')A+(b+xb')B+(c+xc')C+\text{etc.}$
where x *is variable; and* a, b, c, etc., a', b', c', etc. *constants which may be zero; and* A, B, C, etc. *fixed points; then the locus of* P *is a straight line.*

For then

$(\ldots\ldots)P=(aA+bB+cC+\text{etc.})+x(a'A+b'B+c'C+\text{etc.})$

hence, whatever be x, P lies on the line joining the point $aA+bB+cC+\text{etc.}$
with the point $a'A+b'B+c'C+\text{etc.}$
and therefore the locus of P is the right line joining these two fixed points.

Examples. [1]. If a point C′ be taken in any one (as AB) of three indefinite straight lines that intersect in A, B, C, and lines (as C′B′A′) be drawn from the fixed point C′ cutting AC, BC (as in B′, A′), then all the intersections of each pair of lines (as BB′, AA′) drawn from B and A to the points of section (B′, A′) lie in a line which passes through C.

Draw a figure with C′ on AB, B′ on AC, and A′ on BC produced; let BB′, AA′ intersect in P, it is required to find the locus of P.

C′ being a fixed point on AB, we may assume

$(p+q)C'=pA+qB$ (1) where p, q are constants.

B′ being a variable point on AC, we may assume

$(1+x)B'= A +xC$ (2) where x is variable.

Eliminate A between (1) and (2), therefore

$$(p+q)C'-p(1+x)B'=qB-pxC=(q-px)A'$$

or replacing C′ by its value

$$pA+qB-p(1+x)B'=(q-px)A'.$$

Hence $\qquad qB-p(1+x)B'=(q-p-px)P$

replacing B′ by its value

$$(q-p-px)P=qB-pA-pxC.$$

Hence the locus of P is a right line through the points $qB-pA$ and C. We may further remark that if the locus of P cuts AB in C″, then AC′BC″ is a line harmonically divided.

[2]. A line is drawn parallel to the base BC of a

triangle ABC, and the points F, G, where it meets the sides AB, AC, joined to any two fixed points D, E on the base, to find the locus of the intersection P of the joining lines, DF, EG.

Draw a figure in which D, E lie on CB, BC produced, F being a variable point on AB, put

$$(1+x)F=A+xB \quad (1) \text{ where } x \text{ is variable.}$$

Then G dividing CA in the same ratio as F does BA, therefore

$$(1+x)G=A+xC. \qquad (2)$$

Also, if we denote BC by a, and DE by e,

$$e(B-C)=a(D-E) \qquad (3)$$

(1)—(2) gives $(1+x)(F-G)=x(B-C)$

whence, using (3), $e(1+x)(F-G)=xa(D-E)$

transposing $xaD-e(1+x)F=(xa-e-ex)P$

replacing F by its value from (1)

$$(xa-e-ex)P=xaD-eA-exB.$$

This containing the variable x only in the first degree, the locus of P is a right line through A and the point $aD-eB$.

[3]. The base MN of a triangle is given, and the sides meet a fixed line AB parallel to the base in points C, D, such that the ratio of AC to BD is given; find the locus of the vertex L. (Salmon, page 46.)

Let $MN=a$, $AB=b$, $DB=x$, let m=the given ratio, then $AC=mx$, and since MN, ACDB are parallel,

$$\frac{M-N}{a}=\frac{A-B}{b}=\frac{D-B}{x}=\frac{A-C}{mx}=\text{therefore }\frac{C-D}{b-x-mx}$$

whence $M(b-x-mx)-aC=N(b-x-mx)-aD$

or $(b-x-mx-a)P=M(b-x-mx)-aC$

replacing aC by its value.

$$\begin{aligned}(-a+b-x-mx)P&=M(b-x-mx)+mxM-mxN-aA\\ &=M(b-x)-mxN-aA.\end{aligned}$$

This containing the variable x only in the first degree, the locus of P is a right line through the points $bM-aA$ and $M+mN$.

201. *Another way of proving that the locus of a point* $P(x, y, z)$ *is a right line, is to prove that the equation to that locus is of the first degree in* x, y, z. See 173.

Examples. [1]. To find the locus of a point P, such that the perpendicular PD from P on BC is equal to the sum of the perpendiculars PE, PF from P on CA, AB.

Let $$(x+y+z)P=xA+yB+zC$$
take distances from BC, CA, AB, then

$$(x+y+z)PD=x\frac{2\text{ area ABC}}{a} \qquad (1)$$

$$(x+y+z)PE=y\frac{2\text{ area ABC}}{b} \qquad (2)$$

$$(x+y+z)PF=z\frac{2\text{ area ABC}}{c} \qquad (3)$$

(1)—(2)—(3) gives, since $PD=PE+PF$

$$0=\frac{x}{a}-\frac{y}{b}-\frac{z}{c}$$

which is the equation to the required locus, and being of the first degree in x, y, z represents a right line. This line passes through the points $aA+bB$, $aA+cC$, $bB-cC$, that is, through the feet of the bisectors of the interior angles C, B, and of the exterior angle A of the triangle ABC.

[2]. If P be any point on the circumference of a fixed circle, O any fixed point, prove that the locus of the point, in which the tangent at P intersects the line which bisects OP at right angles is a straight line. (Ferrers, page 152.)

Let C be the centre of the circle, let D be any fixed

point, and take OCD for the triangle of reference. Draw OO′, DD′ to touch the circle, radius r.

Let $(x+y+z)R=xO+yC+zD$

be any point on the required locus

Multiply by O^2—circle

then since by question RO=the tangent RP

therefore $0=x(-OO'^2)+y(OC^2+r^2)+z(OD^2-DD'^2)$

or $0=-xOO'^2+y(OC^2+r^2)+z(OD^2-DD'^2)$ is the required equation, which being of the first degree in x, y, z represents a right line.

CHAPTER XII.

TRIANGULAR OR AREAL EQUATION OF A CIRCLE.

202. *To find the general triangular equation of a given circle.*

Let ABC be the triangle of reference. From A, B, C draw tangents AA′, BB′, CC′ to the given circle.

Let $(x+y+z)P=xA+yB+zC$ be any point on this circle. Multiply by the circle. See 153

therefore

$$0=(x+y+z)(xA'A^2+yB'B^2+zC'C^2)-(yza^2+zxb^2+xyc^2)$$

the required equation.

Examples. [1]. The areal equation of the circle circumscribing ABC is $0=yza^2+zxb^2+xyc^2$.

For the tangents from A, B, C to the circumscribed circle all vanish, that is $A'A=0=B'B=C\,C$.

[2]. The areal equation of the circle inscribed in the triangle ABC is

$$0=(x+y+z)\{(s-a)^2x+(s-b)^2y+(s-c)^2z\} \\ -(yza^2+zxb^2+xyc^2)$$

For the tangents from A, B, C to the inscribed circle are $s-a$, $s-b$, $s-c$.

[3]. The areal equation of the escribed circle of the

triangle ABC which touches BC and AB, AC produced is
$0=(x+y+z)\{s^2x+(s-c)^2y+(s-b)^2z\}-(yza^2+zxb^2+xyc^2)$
For the tangents from A, B, C to this circle are $s, s-c, s-b$.

[4]. The areal equation to the nine points circle is

$$0=(x+y+z)\{\tfrac{1}{2}bc\cos A.x+\tfrac{1}{2}ca\cos B.y+\tfrac{1}{2}ab\cos C.z\}$$
$$-(yza^2+zxb^2+xyc^2)$$

For let AA′ be drawn from A to touch it in A′; let F′ be the middle point of AB, and CF perpendicular on AB, then since the nine points circle passes through F and F′

$$A'A^2=AF'.AF=\tfrac{1}{2}c.b\cos A.$$

203. *To find the areal equation of a circle having a given centre* L *and a given radius r.*

Let $(x+y+z)P=xA+yB+zC$ be any point on this circle. Multiply by the circle

hence $0=(x+y+z)\{x(AL^2-r^2)+y(BL^2-r^2)+z(CL^2-r^2)\}$
$-(yza^2+zxb^2+xyc^2)$.

204. *To find the areal equation to a circle, whose centre is the point* O (e, f, g) *and radius r.*

Let P (x, y, z) be any point on the circumference, then $PO^2=r^2$, or See 197

$$D(yg-fz,\ ze-xg,\ xf-ey)=(x+y+z)^2(e+f+g)^2r^2$$

the required equation, of which the expanded form is

$$(yg-fz)^2a^2+(ze-xg)^2b^2+(xf-ey)^2c^2$$
$$-2bc\cos A(ze-xg)(xf-ey)$$
$$-2ca\cos B(xf-ey)(yg-fz)$$
$$-2ab\cos C(yg-fz)(ze-xg)$$
$$=r^2(e+f+g)^2(x+y+z)^2.$$

205. *Circumscribed circle.*

(i.) *To find the areal equation to the circle circumscribing the triangle* ABC.

Let $(x+y+z)P=xA+yB+zC$ (1) be any point on it.
Multiply by the circumscribed circle.

therefore $0=-(yz\text{BC}^2+zx\text{CA}^2+xy\text{AB}^2)$

or $0=yza^2+zxb^2+xyc^2$ (2) the required equation

or $0=\frac{a^2}{x}+\frac{b^2}{y}+\frac{c^2}{z}$

(ii.) The geometrical meaning of this equation, when P lies on the arc BC, is

$$0=\frac{a^2}{-\text{area PBC}}+\frac{b^2}{\text{PCA}}+\frac{c^2}{\text{PAB}} \qquad (3)$$

or if PD, PE, PF be perpendiculars on BC, CA, AB,

$$\frac{a}{\text{PD}}=\frac{b}{\text{PE}}+\frac{c}{\text{PF}}$$

hence, if ABC is equilateral, $\frac{1}{\text{PD}}=\frac{1}{\text{PE}}+\frac{1}{\text{PF}}$

Again, area PBC$=\frac{1}{2}$PB.PC sin A, and equation (3) also gives $0=-a\text{PA}+b\text{PB}+c\text{PC}$

or $a\text{PA}=b\text{PB}+c\text{PC}$

which in an equilateral triangle, reduces to PA=PB+PC.

(iii.) Again, the area DEF, which is generally

$$\frac{1}{2}\frac{(2\text{ area ABC})^3}{a^2b^2c^2(x+y+z)^2}(yza^2+zxb^2+xyc^2) \qquad \text{See 133}$$

here vanishes, see (2)

Therefore the feet of the perpendiculars from any point of the circumscribed circle of the triangle ABC on the sides BC, CA, AB lie in one right line.

206. *To find the equation of a circle, radius m, concentric with the circumscribed circle.*

Let $(x+y+z)\text{P}=x\text{A}+y\text{B}+z\text{C}$ be any point on circle m.

× the circle required,

then $0=(x+y+z)\{x(q^2-m^2)+y(q^2-m^2)+z(q^2-m^2)\}$
$\qquad -(yza^2+zxb^2+xyc^2)$

or $(x+y+z)^2(q^2-m^2)=yza^2+zxb^2+xyc^2.$ (1)

Cor. Let PD, PE, PF be perpendiculars on BC, CA, AB, then generally

$$\text{the area DEF}=\frac{\frac{1}{8}(2\text{ area ABC})^3}{a^2b^2c^2(x+y+z)^2}(yza^2+zxb^2+xyc^2)$$

here, on account of equation (1),

$$\text{the area DEF}=\frac{\frac{1}{8}(2\text{ area ABC})^3}{a^2b^2c^2}(q^2-m^2)=\text{constant}.$$

Therefore: *If from any point* P *of a circle, concentric with the circle about* ABC, *perpendiculars* PD, PE, PF *be drawn on* BC, CA, AB, *the triangle* DEF *is of constant area.*

207. *Inscribed circle.*

(i.) *To find the areal equation to the inscribed circle of the triangle of reference* ABC.

Let $(x+y+z)\mathrm{P}=x\mathrm{A}+y\mathrm{B}+z\mathrm{C}$ be any point on the inscribed circle,

multiply by the incribed circle,

therefore, if D, E, F be the points of contact of BC, CA, AB, the required equation is

$$0=(x+y+z)(x\mathrm{AE}^2+y\mathrm{BF}^2+z\mathrm{CD}^2)-(yza^2+zxb^2+xyc^2)$$

but $\mathrm{AE}=s-a$, $\mathrm{BF}=s-b$, $\mathrm{CD}=s-c$, and the equation becomes

$$0=(x+y+z)\{x(s-a)^2+y(s-b)^2+z(s-c)^2\}$$
$$-(yza^2+zxb^2+xyc^2).$$

(ii.) Which may be written

$$0=x^2(s-a)^2+y^2(s-b)^2+z^2(s-c)^2-2yz(s-b)(s-c)$$
$$-2zx(s-c)(s-a)-2xy(s-a)(s-b)$$

or $$0=\sqrt{x(s-a)}+\sqrt{y(s-b)}+\sqrt{z(s-c)}$$

In this equation one of the radicals must have a different sign from the other two. If P lies on the arc EF, then

$$\sqrt{x(s-a)}=\sqrt{y(s-b)}+\sqrt{z(s-c)}.$$

When ABC is equilateral and P lies on arc EF,

$$\sqrt{x}=\sqrt{y}+\sqrt{z}$$

or $$\sqrt{\text{area PBC}}=\sqrt{\text{PCA}}+\sqrt{\text{PAB}}$$

or if PG, PH, PK be perpendicular on BC, CA, AB,

$$\sqrt{\text{PG}}=\sqrt{\text{PH}}+\sqrt{\text{PK}}.$$

208. *Escribed circle touching* BC.

Its equation is readily proved to be

$$(x+y+z)\{xs^2+y(s-c)^2+z(s-b)^2\}=(yza^2+zxb^2+xyc^2)$$

or $$0=\sqrt{-xs}+\sqrt{y(s-c)}+\sqrt{z(s-b)}.$$

209. *Equation to the circle touching the sides of an isosceles triangle* ABC *at the extremities* B, C *of its base.*

Let $(x+y+z)\text{P}=x\text{A}+y\text{B}+z\text{C}$ be any point on this circle,

multiply by the circle,

hence $0=(x+y+z)xb^2-(yza^2+zxb^2+xyc^2)$

but $c=b$

therefore $0=x^2b^2-yza^2$ is the required equation.

Geometrical meaning of this equation.

Draw PD, PE, PF, perpendiculars on BC, CA, AB, then $\text{PD}^2=\text{PE.PF}$, that is,

If from any point of a circle, perpendiculars be let fall on any two tangents, and on their chord of contact, the square of the last equals the rectangle of the other two.

210. *To find the triangular equation to the nine points circle.*

It is the circle about D, E, F, the bisections of BC, CA, AB, hence P being any point on it,

$$0=\frac{\text{EF}^2}{\text{area PEF}}+\frac{\text{FD}^2}{\text{PFD}}+\frac{\text{DE}^2}{\text{PDE}}$$ See 205

Let $(x+y+z)P=xA+yB+zC$

since $2E=C+A$

and $2F=A+B$

$4(x+y+z)\text{area PEF}=(-x+y+z)ABC$

also $4EF^2=a^2$

hence $0=\dfrac{a^2}{-x+y+z}+\dfrac{b^2}{x-y+z}+\dfrac{c^2}{x+y-z}$

which gives $0=x^2bc\cos A+y^2ca\cos B+z^2ab\cos C$
$-(yza^2+zxb^2+xyc^2)$.

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211. *The equation,*

$$0=x^2a^2+y^2b^2+z^2c^2+2yzbc\cos A+2zxca\cos B+2xyab\cos C$$

represents a circle.

For since $2bc\cos A=b^2+c^2-a^2$, etc.
this equation may be written

$$0=(x+y+z)(xa^2+yb^2+zc^2)-(yza^2+zxb^2+xyc^2)$$

and is the equation to the circle such that the tangents to it from A, B, C are of lengths a, b, c.

212. *Self-conjugate circle.*

(i.) *The equation* $0=x^2\cot A+y^2\cot B+z^2\cot C$ *represents a circle.*

For this equation gives successively

$$0=x^2\frac{\cos A}{a}+y^2\frac{\cos B}{b}+z^2\frac{\cos C}{c}$$

$$0=x^2(b^2+c^2-a^2)+y^2(c^2+a^2-b^2)+z^2(a^2+b^2-c^2)$$

$$0=(x+y+z)\{x(b^2+c^2-a^2)+y(c^2+a^2-b^2)+z(a^2+b^2-c^2)\}$$
$$-yz(a^2+b^2-c^2+c^2+a^2-b^2)-2zxb^2-2xyc^2$$

or $\quad 0=(x+y+z)\ \{xbc\cos A+yca\cos B+zab\cos C\}$
$-(yza^2+zxb^2+xyc^2)$

Therefore the above equation represents a circle such that the squares of the tangents on it from A, B, C are bc cos A, ca cos B, ab cos C.

(ii.) The equation $0=x^2 \cot A+y^2 \cot B+z^2 \cot C$ shows that the circle is real, only when the triangle ABC is obtuse angled.

(iii.) With regard to this circle, A, B, C are the poles of the opposite sides BC, CA, AB.

For put $x=0$, then $y\sqrt{\cot B}=\pm z\sqrt{-\cot C}$, that is, the points in which BC is cut by the circle are $\sqrt{\tan B}.B+\sqrt{-\tan C}.C$ and $\sqrt{\tan B}.B-\sqrt{-\tan C}.C$.

Hence BC is cut harmonically by the circle, therefore the polar of C passes through B; similarly by putting $y=0$, we prove it to pass through A; therefore AB is the polar of C.

Similarly CA is the polar of B, and BC is the polar of A.

(iv.) The circle $0=x^2 \cot A+y^2 \cot B+z^2 \cot C$ is called the self-conjugate circle.

(v.) The squares of the tangents from A, B, C to the self-conjugate circle are double of the squares of the tangents from A, B, C to the nine points circle. See 202.

213. *The circumscribed circle, the nine points circle, and the self-conjugate circle, have a common chord, real or imaginary.*

The equations to those circles are

$$0=yza^2+zxb^2+xyc^2$$

$$0=x^2(b^2+c^2-a^2)+y^2(c^2+a^2-b^2)+z^2(a^2+b^2-c^2)$$
$$-2(yza^2+zxb^2+xyc^2)$$

$$0=x^2(b^2+c^2-a^2)+y^2(c^2+a^2-b^2)+z^2(a^2+b^2-c^2).$$

Any one of these three equations may be derived

from the other two, hence the three circles pass through the same two points, real or imaginary.

214. *To find the conditions that the general equation of the second degree,*

$$0=lx^2+my^2+nz^2+2pyz+2qzx+2rxy$$

should represent a circle.

Let e, f, g be the tangents from A, B, C to this circle, then its equation is also

$$0=x^2e^2+y^2f^2+z^2g^2+yz(f^2+g^2-a^2)+zx(g^2+e^2-b^2)$$
$$+xy(e^2+f^2-c^2).$$

Comparing the two equations, and h being some constant, we have

$$e^2=hl$$
$$f^2=hm$$
$$g^2=hn$$
$$f^2+g^2-a^2=2hp$$
$$g^2+e^2-b^2=2hq$$
$$e^2+f^2-c^2=2hr.$$

The required conditions must be the results of the elimination of e, f, g, h between these six equations.

The elimination of e^2, f^2, g^2 gives

$$hm+hn-a^2=2hp$$
$$hn+hl-b^2=2hq$$
$$hl+hm-c^2=2hr$$

whence, eliminating h, the required conditions are

$$\frac{m+n-2p}{a^2}=\frac{n+l-2q}{b^2}=\frac{l+m-2r}{c^2}$$

215. *To determine the intersections of a circle with the line at infinity.*

Let p, q, r denote the tangents from A,B,C to the circle, then its equation is

$$0=(x+y+z)(xp^2+yq^2+zr^2)-(yza^2+zxb^2+xyc^2).$$

The line at infinity is $0=x+y+z$.

At the points where the line at infinity intersects the circle, the two equations are simultaneous; hence also for those two points,

$$0=yza^2+zxb^2+xyc^2.$$

Or the line at infinity intersects any circle on the circumscribed circle; that is, *all circles intersect the line at infinity in the same two points*, which are, of course, imaginary. They are called *the two circular points at infinity.*

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216. *To find the equation to the tangent at any point* E *on a circle.*

Let $$(x+y+z)\mathrm{P}=x\mathrm{A}+y\mathrm{B}+z\mathrm{C} \qquad (1)$$ be any point on the tangent at E, and let AA′, BB′, CC′ be the tangents from A, B, C to the circle,

then, multiplying (1) by E^2— the circle, since PE is tangent to the circle, therefore the required equation is

$$0=x(\mathrm{AE}^2-\mathrm{A'A}^2)+y(\mathrm{BE}^2-\mathrm{B'B}^2)+z(\mathrm{CE}^2-\mathrm{C'C}^2).$$

Corollary 1. Let the tangent at E meet BC in F, then $(\ldots\ldots)\mathrm{F}=(\mathrm{CE}^2-\mathrm{C'C}^2)\mathrm{B}-(\mathrm{BE}^2-\mathrm{B'B}^2)\mathrm{C}$.

Hence: *If the tangent at any point* E *on any circle meets any line* BC *in* F, *and if* BB′, CC′ *be tangents to the circle,*

$$\mathrm{BE}^2-\mathrm{BB'}^2 : \mathrm{CE}^2-\mathrm{C'C}^2=\mathrm{BF} : \mathrm{CF}.$$

Let BE, CE meet the circle again in B″, C″, then $\mathrm{BB'}^2=\mathrm{BE.BB''}$, and the above proportion becomes

$$\mathrm{BE.EB''} : \mathrm{CE.EC''}=\mathrm{BF} : \mathrm{CF}.$$

If B, C lie on a circle centre E, then the proportion becomes $$\mathrm{BF} : \mathrm{CF}=\mathrm{EB''} : \mathrm{EC''}.$$

Corollary 2. (i.) *The tangent at the point* E *on the circumscribed circle is*

$$0=x\text{AE}^2+y\text{BE}^2+z\text{CE}^2.$$

(ii.) If it meets BC in F, then

$$\text{BF}:\text{CF}=\text{BE}^2:\text{CE}^2.$$

(iii.) If tangents at the points E, E′ of a circle intersect on a chord BC produced, then

$$\text{BE}:\text{BE}'=\text{CE}:\text{CE}'.$$

(iv.) If the tangent at any point E of a circle meets the sides of an inscribed triangle ABC in A′, B′, C′ respectively; and if BB′, CC′ intersect in A″; CC′, AA′ in B″; AA′, BB′ in C″; then AA″, BB″, CC″ meet in the point $\dfrac{\text{A}}{\text{AE}^2}+\dfrac{\text{B}}{\text{BE}^2}+\dfrac{\text{C}}{\text{CE}^2}$

Corollary 3. The tangent at any point E on the inscribed circle is

$$0=x\{\text{AE}^2-(s-a)^2\}+y\{\text{BE}^2-(s-b)^2\}+z\{\text{CE}^2-(s-c)^2\}$$

Let it cut BC in F, then

$$\text{BF}:\text{CF}=\text{BE}^2-(s-b)^2:\text{CE}^2-(s-c)^2.$$

Or if a circle touches BC in L, and any tangent to it at E cut BC in F, then

$$\text{BF}:\text{CF}=\text{BE}^2-\text{BL}^2:\text{CE}^2-\text{CL}^2.$$

Corollary 4. If two circles touch in E; and AA′, BB′, CC′ be tangents from A, B, C to one of them; and AA″, BB″, CC″ tangents from A, B, C to the other, then

$$\text{AE}^2-\text{AA}'^2:\text{BE}^2-\text{BB}'^2:\text{CE}^2-\text{CC}'^2$$
$$=\text{AE}^2-\text{AA}''^2:\text{BE}^2-\text{BB}''^2:\text{CE}^2-\text{CC}''^2.$$

217. *To find the length* PT *of the tangent from a given point to a given circle.*

Let $\qquad (x+y+z)\text{P}=x\text{A}+y\text{B}+z\text{C} \qquad (1)$

be the given point, from which the tangent PT is drawn

to the given circle, and let AA′, BB′, CC′ be tangents from A, B, C to this circle,

multiply (1) by the circle, therefore

$$(x+y+z)^2\mathrm{PT}^2=(x+y+z)(x\mathrm{AA}'^2+y\mathrm{BB}'^2+z\mathrm{CC}'^2)$$
$$-(yza^2+zxb^2+xyc^2).$$

Example. If from any point tangents, real or imaginary, be drawn to the inscribed and escribed circles, the sum of the squares of these tangents exceeds four times the square of the tangent from the same point to the circumscribed circle by a constant quantity.

Let PI, PE_1, PE_2, PE_3, PK be tangents from the point $\mathrm{P}(x, y, z)$ to the inscribed, escribed, and circumscribed circles of ABC, then

$$(x+y+z)^2\mathrm{PI}^2=(x+y+z)\{x(s-a)^2+y(s-b)^2+z(s-c)^2\}$$
$$-(yza^2+zxb^2+xyc^2)$$

and $$(x+y+z)^2\mathrm{PK}^2=-(yza^2+zxb^2+xyc^2)$$

hence

$$(x+y+z)(\mathrm{PI}^2-\mathrm{PK}^2)=x(s-a)^2+y(s-b)^2+z(s-c)^2$$

Similarly

$$(x+y+z)(\mathrm{PE}_1{}^2-\mathrm{PK}^2)=xs^2+y(s-c)^2+z(s-b)^2$$

and

$$(x+y+z)(\mathrm{PE}_2{}^2-\mathrm{PK}^2)=x(s-c)^2+ys^2+z(s-a)^2$$

and

$$(x+y+z)(\mathrm{PE}_3{}^2-\mathrm{PK}^2)=x(s-b)^2+y(s-a)^2+zs^2$$

add and divide by $x+y+z$, therefore

$$\mathrm{PI}^2+\mathrm{PE}_1{}^2+\mathrm{PE}_2{}^2+\mathrm{PE}_3{}^2-4\mathrm{PK}^2$$
$$=s^2+(s-a)^2+(s-b)^2+(s-c)^2$$
$$=a^2+b^2+c^2$$

The same result is more readily obtained by multiplying $(x+y+z)\mathrm{P}=x\mathrm{A}+y\mathrm{B}+z\mathrm{C}$ by

circle O + circle O_1 + circle O_2 + circle O_3
—4 circumscribed circle.

Corollary. The sum of the squares of the tangents drawn from any point of the circumscribed circle to the inscribed and escribed circles is always equal to $a^2+b^2+c^2$.

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218. *To find the equation of the polar of any point* E *with regard to a circle, centre* D, *radius* r,

The polar of E is a line PF perpendicular on DE, and intersecting DE at a point F, such that $DE.DF=r^2$ hence $$DE^2.F=r^2E+(DE^2-r^2)D \quad (1)$$

Let $$(x+y+z)P=xA+yB+zC \quad (2)$$
be any point on the required polar, then
$$(P-F)(D^2-E^2)=0$$
substituting for P and F their values from (1) and (2), and reducing, we get for the required equation
$$(x+y+z)(2r^2-DE^2)$$
$$=x(AD^2-AE^2)+y(BD^2-BE^2)+z(CD^2-CE^2)$$
The equation to the polar is also
$$(x+y+z)E'E^2=(x+y+z)(DE^2-r^2)$$
$$=x(AE^2-A'A^2)+y(BE^2-B'B^2)+z(CE^2-C'C^2)$$
where $A'A$, $B'B$, $C'C$, $E'E$ are tangents from A, B, C, E to the circle.

Corollary. (i.) The equation to the polar of A with regard to any circle is
$$(x+y+z)AA'^2=x(0-AA'^2)+y(BA^2-BB'^2)$$
$$+z(CA^2-CC'^2)$$
or $$0=2xAA'^2+y(AA'^2+BB'^2-AB^2)$$
$$+z(CC'^2+AA'^2-CA^2).$$

(ii.) Hence, if this polar meets BC in F, we have $BF:CF=AA'^2+BB'^2-AB^2:CC'^2+AA'^2-CA^2.$

(iii.) Let F falls on C, that is let C be a point on the polar of A, then $CC'^2+AA'^2=CA^2$.

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219. *To find the equation to the radical axis of two circles centres* D, E, *radii* d, e.

The radical axis is the locus of the point P, such that the tangents PT, PQ from it to the circles D, E are equal.

Let $(x+y+z)P=xA+yB+zC$, multiply by D^2-E^2 therefore

$$(x+y+z)(PD^2-PE^2)$$
$$=x(AD^2-AE^2)+y(BD^2-BE^2)+z(CD^2-CE^2)$$

but

$$PD^2-PE^2=(PD^2-d^2)-(PE^2-e^2)+d^2-e^2$$
$$=PT^2-PQ^2+d^2-e^2=d^2-e^2$$

since $PT=PQ$; hence the required equation is

$$(x+y+z)(d^2-e^2)$$
$$=x(AD^2-AE^2)+y(BD^2-BE^2)+z(CD^2-CE^2)$$

This equation may be written

$$0=x\{(AD^2-d^2)-(AE^2-e^2)\}+y\text{ etc.}+z\text{ etc.}$$

or

$$0=x(AA'^2-AA''^2)+y(BB'^2-BB''^2)$$
$$+z(CC'^2-CC''^2) \qquad (1)$$

where AA′, BB′, CC′, are tangents from A, B, C to the circle D; and AA″, BB″, CC″ to the circle centre E.

But the equations of these circles are

$$0=(x+y+z)(xAA'^2+yBB'^2+zCC'^2)$$
$$-(yza^2+zxb^2+xyc^2) \qquad (2)$$
$$0=(x+y+z)(xAA''^2+yBB''^2+zCC''^2)$$
$$-(yza^2+zxb^2+xyc^2) \qquad (3)$$

hence the equation (1) being derived from (2) and (3) by subtraction represents a locus through the intersec-

tions of (2) and (3). Therefore the radical axis passes through the intersections of (2) and (3), and therefore, when those intersections are real, the radical axis is the common chord; when the circles touch, the radical axis is the common tangent.

220. *The radical axes of three circles meet in a point called the radical centre of the three circles.*

Let $(e, f, g)(e', f', g')(e'', f'', g'')$ be the tangents from A, B, C to the three circles centres O, O′, O″.

the radical axis of O′, O″ is

$$0=x(e'^2-e''^2)+y(f'^2-f''^2)+z(g'^2-g''^2)$$

That of O″, O is

$$0=x(e''^2-e^2)+y(f''^2-f^2)+z(g''^2-g^2)$$

and that of O, O′

$$0=x(e^2-e'^2)+y(f^2-f'^2)+z(g^2-g'^2)$$

Any one of these three equations being derivable from the other two by subtraction, the three radical axes meet in one point.

Examples. [1]. If e, f, g be the tangents from A, B, C to a circle, the radical axis of this circle and the circumscribed circle is $\quad 0=xe^2+yf^2+zg^2$

$e^2 : f^2 : g^2$ as the perpendiculars from A, B, C on this radical axis. See 172.

[2]. D being any point, $0=x\text{AD}^2+y\text{BD}^2+z\text{CD}^2$ is the equation of a right line, locus of P, such that PD equals the tangent from P to the circumscribed circle.

[3]. The radical axis of the circumscribed and inscribed circle is $0=x(s-a)^2+y(s-b)^2+z(s-c)^2$.

[4]. The radical axis of the circumscribed and of the escribed circle touching BC is

$$0=xs^2+y(s-c)^2+z(s-b)^2.$$

If P be any point on this radical axis, and ABC be equilateral, then 8 area PBC=area ABC.

[5]. If P, Q, R be the points in which the radical axis of the circumscribed and inscribed circles meet BC, CA, AB; and P', Q', R'; P'', Q'', R''; P''', Q''', R''', the like points for the radical axes of the circumscribed and escribed circles; then P', Q'', R''' lie in one right line; so are P, Q''', R''; Q, P''', R'; R, Q', P''; and the intersections of $R''\,Q'''$ with BC, $R'\,P'''$ with CA, $Q'\,P''$ with AB, all lie in the radical axis of the circumscribed and inscribed circles.

[6]. The radical axis of the inscribed circle and of the escribed circle touching BC is

$$0=x(b+c)+(y-z)(b-c).$$

[7]. The radical axis of the escribed circles touching BC, CA is $0=(x-y)(a+b)+z(a-b)$.

[8]. The radical centre of the circles O, O_1, O_2 is the point $b-c$, $a-c$, $a+b$.

For it is the intersection of the lines

$$0=x(b+c)+y(b-c)+z(c-b)\quad\text{See Example [6].}$$
$$0=x(a+b)+y(-a-b)+z(a-b)$$

Therefore the required radical centre is the point

$$(b-c)(a-b)+(a+b)(c-b),(c-b)(a+b)-(b+c)(a-b),$$
$$-(b+c)(a+b)-(a+b)(b-c)$$

or the point $\quad b-c,\ a-c,\ a+b.$

[9]. The radical axis of O_2, O_3 is

$$0=x(b-c)+(y-z)(b+c).$$

[10]. The radical centre of the three escribed circles is the point $b+c$, $c+a$, $a+b$; that is, the centre of gravity of the perimeter of the triangle ABC.

Therefore the tangents from this point to the three escribed circles are equal.

[11]. The radical axis common to the circumscribed circle, the nine points circle, and the self-conjugate circle, is $0=x\cot A+y\cot B+z\cot C$.

[12]. The common tangent, or radical axis, of the nine points circle and inscribed circle is

$$0=\frac{x}{b-c}+\frac{y}{c-a}+\frac{z}{a-b}$$

[13]. The common tangent, or radical axis, of the nine points circle and escribed circle touching BC is

$$0=\frac{x}{b-c}+\frac{y}{c+a}-\frac{z}{a+b}$$

[14]. The mean point of the four radical centres of the inscribed and escribed circles is the centre of the nine points circle.

For if R, R_1, R_2, R_3 be the radical centres of

$$(O_1O_2O_3),\ (O_2O_3O),\ (O_3OO_1),\ (OO_1O_2),$$

we have, see Example [8],

$$2(a+b-c)R_3=(b-c)A+(a-c)B+(a+b)C$$
$$=(a+b-c)(A+B+C)-(aA+bB-cC)$$

hence $2R_3=3G-O_3$ (1)

similarly $2R_2=3G-O_2$ (2)

and $2R_1=3G-O_1$ (3)

Again

$$2(a+b+c)R=(b+c)A+(c+a)B+(a+b)C \text{ See ex. [10]}$$
$$=(a+b+c)(A+B+C)-(aA+bB+cC)$$

hence $2R=3G-O$ (4)

adding $2(R+R_1+R_2+R_3)$

$$=12G-(O+O_1+O_2+O_3)=12G-4Q. \text{ See 54}$$

$R+R_1+R_2+R_3=6G-2Q=4Q'$, where Q' is the centre of the nine points circle. See 61.

[15]. The radical centre of the nine points circle,

inscribed circle, and first escribed circle is the point $(b-c)^2, c^2-a^2, b^2-a^2$.

General Equation of a Circle.

221. *To find the centre of the circle.*

$$0=(x+y+z)(xe+yf+zg)-(yza^2+zxb^2+xyc^2).$$

Let the centre be $(l+m+n)O=lA+mB+nC$, (1) and let r denote the radius. Since e, f, g are the squares of the tangents from A, B, C to the circle, therefore

$$e=AO^2-r^2$$
$$f=BO^2-r^2$$
$$g=CO^2-r^2$$

hence $$f-g=BO^2-CO^2 \qquad (2)$$

Multiply (1) by B^2-C^2, and make use of (2); then

$$(l+m+n)(f-g)=l(c^2-b^2)+m(0-a^2)+n(a^2-0)$$

similarly

$$(l+m+n)(g-e)=l(b^2-0)+m(a^2-c^2)+n(0-b^2)$$

whence $8(\text{area } ABC)^2O$

$$=a\{abc\cos A-b\cos C(e-f)-c\cos B(e-g)\}A$$
$$+b\{abc\cos B-c\cos A(f-g)-a\cos C\,(f-e)\}B \quad (3)$$
$$+c\{abc\cos C-a\cos B(g-e)-b\cos A\,(g-f)\}C$$

Examples. [1]. But

$$\text{area } ABC.O=OBC.A+OCA.B+OAB.C$$

hence 8 area ABC . area OBC

$$=a\{abc\cos A-b\cos C(e-f)-c\cos B(e-g)\}$$

[2]. Thus: If AT touches in T a circle on BC as diameter, $AT^2=bc\cos A$.

[3]. Again: If ABC be an equilateral triangle, and AA′, BB′, CC′ tangents from A, B, C to a circle having its centre on BC, then $BC^2=2A'A^2-B'B^2-C'C^2$.

[4]. The centres of all circles, for which the tangents from A, B, C are equal, coincide with the centre of the circumscribed circle.

[5]. The centre of the self-conjugate circle of ABC is the intersection P of the perpendiculars of this triangle. See 26.

For this circle, $e=bc \cos \mathrm{A}, f=ca \cos \mathrm{B}, g=ab \cos \mathrm{C}$. Substitute in (3).

[6]. The centre of the circle, to which the tangents from A, B, C are of lengths a, b, c, is

$$a(\cos \mathrm{A}-\cos \mathrm{B} \cos \mathrm{C})\mathrm{A}+\text{etc. } \mathrm{B}+\text{etc. } \mathrm{C};$$

that is, the point 2Q—P, see 31 and 26; and therefore the line joining this centre with the centre P of the self-conjugate circle is bisected by the centre Q of the circumscribed circle.

222. *To find the radius r of the circle.*

$$0=(x+y+z)(xe+yf+zg)-(yza^2+zxb^2+xyc^2).$$

The equation (3) reduces to

$$\begin{aligned}4(\text{area ABC})\mathrm{O}=&\{a^2 \cot \mathrm{A}-(e-f)\cot \mathrm{C}-(e-g)\cot \mathrm{B}\}\mathrm{A}\\ &+\{b^2 \cot \mathrm{B}-(f-g)\cot \mathrm{A}-(f-e)\cot \mathrm{C}\}\mathrm{B} \quad (4)\\ &+\{c^2 \cot \mathrm{C}-(g-e)\cot \mathrm{B}-(g-f)\cot \mathrm{A}\}\mathrm{C}\end{aligned}$$

multiply by the circle itself, observe that

$$4 \text{ area ABC}=a^2 \cot \mathrm{A}+b^2 \cot \mathrm{B}+c^2 \cot \mathrm{C},$$

reduce, change signs, then

$$\begin{aligned}16(\text{area ABC})^2r^2=&\{a^2b^2c^2\\ &-2abc(a \cos \mathrm{A}e+b \cos \mathrm{B}f+c \cos \mathrm{C}g)\\ &+(a^2e^2+b^2f^2+c^2g^2-2bc \cos \mathrm{A}fg-2ca \cos \mathrm{B}ge\\ &\qquad -2ab \cos \mathrm{C}ef)\}. \quad (5)\end{aligned}$$

Examples. [1]. The square of the radius of the self-conjugate circle is $-4q^2 \cos \mathrm{A} \cos \mathrm{B} \cos \mathrm{C}$.

For this circle

$$e=bc \cos \mathrm{A}, f=ca \cos \mathrm{B}, g=ab \cos \mathrm{C}.$$

The self-conjugate circle is real only when the triangle ABC is obtuse angled; its centre is the intersection P of the perpendiculars of ABC; and if PA meets BC in D, its radius is the mean proportional between PA, PD. The self-conjugate circle passes through the points of contact of the tangents from P to any circle through A and D.

[2]. The radius of the circle, to which the tangents from A, B, C are of lengths a, b, c, is double of the radius of the self-conjugate circle. The line joining the centres of these two circles is bisected by the centre of the circumscribed circle.

[3]. Trigonometrical formulæ can be obtained from the equation (5); for iustance, by writing

$$\frac{\text{area ABC}}{s}, (s-a)^2, (s-b)^2, (s-c)^2, \text{ for } r, e, f, g.$$

By writing 0, q^2, q^2, q^2, for r, e, f, g, we get

$$q^2 = \frac{\frac{1}{2}abc}{a \cos A + b \cos B + c \cos C}$$

[4]. From the equation (4)

$$4 \text{ area OBC} = a^2 \cot A - (e-f)\cot C - (e-g)\cot B,$$

where O is the centre of a circle, to which the squares of the tangents from A, B, C are e, f, g.

Distances of a Point from the Points A, B, C.

223. Since a point may be considered as a circle, whose radius vanishes, therefore

If e, f, g denote the squares of the distances of a point O *from the points* A, B, C, *then*, see 222,

4(area ABC)O

$$=\{a^2\cot A-(e-f)\cot C-(e-g)\cot B\}A$$
$$+\{b^2\cot B-(f-g)\cot A-(f-e)\cot C\}B \quad (4)$$
$$+\{c^2\cot C-(g-e)\cot B-(g-f)\cot A\}C.$$

and 4area $OBC=a^2\cot A-(e-f)\cot C-(e-g)\cot B.$ (6)

and $0=a^2b^2c^2-2abc(ae\cos A+bf\cos B+cg\cos C)$
$$+(a^2e^2+b^2f^2+c^2g^2-2bc\cos A\,fg-2ca\cos B\,ge$$
$$-2ab\cos C\,ef.) \quad (7)$$

Examples. [1]. D being the middle point of BC $\cot A:\cot B+\cot C=AD^2-\frac{1}{4}BC^2:BC^2$. Use (6)

[2]. If the line bisecting AB at right angles meets BC in R, $BC^2\cot A=(RB^2-RC^2)\cot B$. Use (6)

[3]. G being the centre of gravity of the triangle ABC,

$\frac{4}{3}$area ABC

$$=a^2\cot A-(GA^2-GB^2)\cot C-(GA^2-GC^2)\cot B$$
$$=b^2\cot B-(GB^2-GC^2)\cot A-(GB^2-GA^2)\cot C$$
$$=c^2\cot C-(GC^2-GA^2)\cot B-(GC^2-GB^2)\cot A.$$

[4]. If the sides of the triangle ABC subtend equal angles at point R,

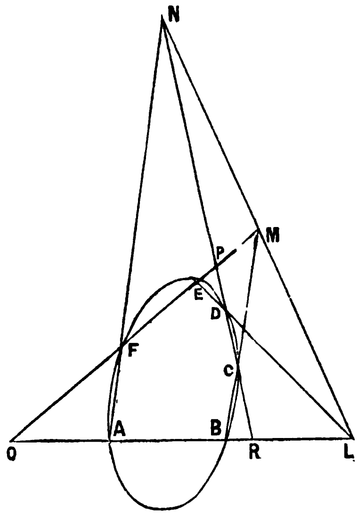
$RB.RC\sqrt{3}$

$$=a^2\cot A-(RA^2-RB^2)\cot C-(RA^2-RC^2)\cot B.$$

[5]. Only when the triangle ABC is right angled, is there a point at distances a, b, c from A, B, C. Then if A be the right angle and ABDC a rectangle, D is the point. Use (7)

PASCAL'S HEXAGON.

224. *The opposite sides* (AB, DE), (BC, EF), (CD, FA) *of a hexagon inscribed in a conic meet in points* L, M, N *lying in one right line.*



Let P, Q, R be the intersections of (CD, EF), (EF, AB), (AB, CD). Through a point O which is not on the conic, draw chords A′B′, C′D′, E′F′, parallel to AB, CD, EF, then

$$PC.PD : PE.PF = OC'.OD' : OE'.OF'$$
$$QE.QF : QA.QB = OE'.OF' : OA'.OB'$$
$$RA.RB : RC.RD = OA'.OB' : OC'.OD'$$

Hence, multiplying and reducing,

$$PC.PD.QE.QF.RA.RB = PE.PF.QA.QB.RC.RD \quad (1)$$

Now $\qquad PR.D = PD.R + RD.P$

and $\qquad PQ.E = PE.Q + QE.P$

Eliminate P, then

$$(PD.QE - PE.RD)L = PD.QE.R - PE.RD.Q \quad (2)$$

similarly

$$(PC.RB - QB.RC)M = PC.RB.Q - QB.RC.P \quad (3)$$

and

$$(QF.RA - QA.PF)N = QF.RA.P - QA.PF.R \quad (4)$$

Between (2) and (3) eliminate Q, then

$$PC.RB(PD.QE - RD.PE)L + RD.PE(PC.RB - QB.RC.)M$$
$$= PC.PD.QE.RB.R - PE.QB.RC.RD.P \quad (5)$$
$$= PC.PD.QE.QF.RA.RB\left(\frac{R}{QF.RA} - \frac{P}{PF.QA}\right)$$
$$= \frac{PC.PD.QE.QF.RA.RB}{PF.QA.QF.RA}(QA.PF - RA.QF)N, \text{see} (1)(4)$$

Hence, L, M, N lie in one right line.

Corollary 1. If each of two consecutive sides of a hexagon inscribed in a conic be parallel to the opposite sides, the two remaining sides are also parallel to each other.

For if AB, BC be parallel to DE, EF ; then L, M are removed to infinity, and therefore also N, hence CD is parallel to AF.

Otherway. The condition that ED should be parallel to AB or QR is $\frac{PD}{RD} = \frac{PE}{QE}$, the condition for the parallelism

of BC, EF is $\frac{RB}{QB}=\frac{RC}{PC}$. Hence, if AB, BC be parallel to DE, EF, then PC.PD.QE.RB=PE.QB.RC.RD whence from (1) QF.RA=PF.QA, or $\frac{QF}{PF}=\frac{QA}{RA}$; that is, AF is parallel to PR, and therefore to CD.

Corollary 2. If two opposite sides CD, FA of a hexagon inscribed in a conic be parallel, then the line joining the intersections L, M of the other opposite sides is parallel to CD and FA.

By hypothesis $\frac{QA}{RA}=\frac{QF}{PF}$, whence from (1), the coefficients of R and P in (5) are equal, and therefore LM is parallel to RP, that is to CD and AF.

THE END.

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ERRATA.

Page 7, 4th line from bottom, a at A, instead of a at B.
Page 26, 10th line from bottom, $-x$A, instead of $-z$A.
Page 55, 12th line from bottom,

$$\frac{b^2}{y}\mathrm{B} + \frac{c^2}{z}\mathrm{C}, \text{ instead of } \frac{b^2}{x}\mathrm{B} + \frac{c^2}{x}\mathrm{C}.$$

Page 71, 12th line from bottom, $y+z$ instead of $z+z$.
Page 88, 2nd line from bottom,

$$\frac{c}{q}\mathrm{CS}, \text{ etc., instead of } \frac{c}{q}\mathrm{CS}.$$

l.